Supplementary Results for Pycia and Woodward [2020]

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In this paper we provide supplementary results and applications for Pycia and Woodward [2020], henceforth, "PW". We omit the model, notation, and assumptions, and refer the interested reader to Section 2 of PW.

1 Equilibrium in Uniform-Price Auctions

1.1 Robust Equilibrium Selection

In the uniform-price auction, equilibrium bidding strategies are unique when the support of supply is sufficiently large (cf. Klemperer and Meyer [1989]). For a similar argument to hold in this context, it is sufficient that the support of supply is $[0, \overline{Q}]$, where $\overline{Q} \geq \sup_s nv^{-1}(R; s)$. Because any supply distribution F can be nearly costlessly transformed to a distribution with unbounded supply—supply can follow distribution F with probability $1-\varepsilon$ and an unbounded distribution with probability ε—we model large supply by assuming that supply is unbounded. We define robust uniform-price bids to be those corresponding to an unbounded distribution of supply. Since equilibrium in the uniform-price auction is ex post, these bids are equilibria given any other distribution of supply and the same reserve price.

Robust uniform-price bids are a natural limit of slight uncertainty about aggregate supply. We define a bid profile to be robust to uncertainty if small changes in the distribution of quantity result in small changes to equilibrium bids.

Definition 1. [Robust Bids] Given supply distribution F and reserve price R, a bid profile $(b^i)_{i=1}^n$ is *robust to uncertainty* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any supply distribution $\tilde{F}$ with $\sup_{Q\in\mathbb{R}} |F(Q) - \tilde{F}(Q)| < \delta$, all equilibrium bid profiles $(\tilde{b}^i)_{i=1}^n$ are such

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that $\sup_{s,q\in[0,\tilde{Q}^R(s)]} |b^i(q;s) - \tilde{b}^i(q;s)| < \varepsilon$ for all bidders i, where[1]

$$\tilde{Q}^R(s) = \min\left\{\max \operatorname{Supp}_F \frac{1}{n}Q, \max \operatorname{Supp}_{\tilde{F}} \frac{1}{n}Q, v^{-1}(R;s)\right\}.$$

Proposition 1. [Bids Robust to Uncertainty] *The unique pay-as-bid equilibrium bid profile is robust to uncertainty. The unique uniform-price equilibrium bid profile that is robust to uncertainty is given by:*

$$b(q;s) = \left(\frac{q}{\hat{Q}^R(s)}\right)^{n-1} R + (n-1)\int_q^{\hat{Q}^R(s)} \left(\frac{q}{x}\right)^{n-1} \frac{v(x;s)}{x} dx, \tag{1}$$

where $\hat{Q}^R(s) = v^{-1}(R;s)$.

We refer to the above uniform-price bid profile as *robust uniform-price bids*.

Proof. Equilibrium bids in the pay-as-bid auction are robust to uncertainty because the bid form given in Theorem 3 of PW is continuous, with respect to the supremum norm, in supply distribution. With regard to the uniform price auction, note that given any supply distribution F there is an unbounded supply distribution $\tilde{F}$ with $\sup_{Q\in\mathbb{R}} |F(Q) - \tilde{F}(Q)| < \varepsilon$. With unbounded supply, robust uniform-price bids are the unique bidding equilibrium in the uniform-price auction [Klemperer and Meyer, 1989]. It follows that for a bid profile to be robust given supply distribution F and reserve price R, it must be such that $\sup_{s,q\in[0,\hat{Q}(s)]} |b^i(q;s) - \tilde{b}^i(q;s)| < \delta$ for any $\delta > 0$ and all bidders i, where $\tilde{b}^i(\cdot;s)$ is the robust uniform-price bid when bidder type is s. Then $(b^i)_{i=1}^n$ is a robust uniform-price bid profile. □

Robust uniform-price bids are continuous, differentiable, strictly below marginal values for all $q \in (0, \hat{Q}(s))$, and equal to marginal values for $q \in \{0, \hat{Q}(s)\}$. No matter which auction format is employed, optimal supply $Q^\star > 0$. In the pay-as-bid design game the optimal deterministic quantity must be binding for some bidder types, $Q^{\star\text{PAB}} < \sup_s v^{-1}(R;s)$, provided the value space is rich. Since robust uniform-price bids are strictly below value on $(0, Q^{\star\text{PAB}}/n]$ for all types s such that $Q^{\star\text{PAB}} < v^{-1}(R;s)$, the pay-as-bid auction generates strictly greater revenue than the uniform-price auction with robust bidding.

Proposition 2. [Strict Dominance of Pay-as-Bid Revenue] *The pay-as-bid design game generates strictly greater revenue than the unique equilibrium of the uniform-price*

[1] Note the distinction between $Q^R(Q,s)$ and $\tilde{Q}^R(s)$: the former is the minimum of Q and aggregate demand at price R when bidders have common signal s. The latter is the minimum upper bound of the supports of the distributions F and $\tilde{F}$, or aggregate demand at R when bidders have common signal s.

design game in robust bids.

In light of Proposition 2 and our main text analysis, combining seller-optimal equilibrium bids on-path and robust bids as off-path threats allows us to construct a variety of PBEs: if the auctioneer implements are particular supply distribution and reserve price, bidders will play the (conditional) seller-optimal equilibrium, and otherwise bidders will play the unique equilibrium in robust uniform-price bids. This implies that any approximately optimal supply distributions and reserve prices can be implemented in a perfect Bayesian equilibrium, as shown in Theorem 9 of PW.

1.2 Price Range in Uniform Price

Robust uniform-price bids provide a natural off-path equilibrium to ensure the selection of a bidder-preferred parameterization of the uniform-price auction format (or any other parameterization consistent ith Theorem 9 of PW). Although robust bidding equilibria remain equilibria when supply is deterministic, restricting attention to deterministic supply allows for a significant range of market clearing prices:

any market clearing price between the reserve price R and the marginal value for the per-capita quantity $v(Q/n;s)$ can obtain in equilibrium.

Proposition. [Range of Prices in Uniform-Price Design Game] *Let $p^\star(s)$ denote the market-clearing price in an equilibrium of the uniform-price design game with deterministic supply Q. Then for all signals s, $p^\star(s) \in [R, \max\{R, v^{-1}(Q/n;s\}]$. Furthermore, for any $p^\star$ such that $p^\star(s) \in [R, \max\{R, v^{-1}(Q/n;s)\}]$ for all s, there is an equilibrium of the uniform-price design game with market-clearing price $p^\star$.*

Proof. To prove the first claim note that $p^\star(s) \geq R$ by definition. If $p^\star(s) > v^{-1}(Q/n;s) > R$, some agent is allocated q_i such that $v(q_i;s) < p^\star(s)$. If she bids $b' = v(\cdot;s)$ instead, she is awarded all units she values above $p^\star(s)$, and possibly more, at a price no greater than $p^\star(s)$. Then she obtains a positive margin on all units received, and possibly decreases her payment; this deviation is profitable.

To prove the second claim, let $p : \operatorname{Supp} s \to [R, \max\{R, v^{-1}(Q/n;s)\}]$ map bidder signals to putative market-clearing prices, and define bids by

$$b\left(q;s,p\right) = \begin{cases} v\left(0;s\right) & \text{if } q < \frac{1}{n}Q, \\ p\left(s\right) & \text{otherwise.} \end{cases}$$

Then downward deviations yield zero quantity, and upward deviations increase the market price without increasing the allocation. Then $(b)_{i=1}^{n}$ is an equilibrium that yields market clearing price $p(s)$. $\square$

1.3 Transparency in Uniform-Price Auctions

Our main results show that in the pay-as-bid auction, optimal supply is transparent (Theorem 5 of PW). These results cannot be directly extended to the uniform-price format, because equilibrium multiplicity allows bidders to select qualitatively different equilibria given different parameterizations of the auction. We show now that when the possibility of equilibrium selection is removed, transparency applies as well to the uniform-price format.

Proposition 3. [Transparency in Uniform-Price Auction] *Suppose that given any supply distribution F and reserve price R, bidders play the equilibrium in robust uniform-price bids. Then optimal supply is deterministic.*

Proof. Applying Proposition 1, we have

$$\begin{aligned}\mathbb{E}\left[\pi^{\mathrm{UP}}\right] &= \mathbb{E}_Q\mathbb{E}_s\left[Qp\left(Q;s\right)\right]\\ &= \mathbb{E}_Q \underbrace{\int_{\mathrm{Supp}\,\sigma}\left[\frac{R}{\hat{Q}\left(s\right)^{n-1}}+\left(n-1\right)\int_Q^{\hat{Q}^R(s)}\frac{v\left(x;s\right)}{x^n}dx\right]\left(\frac{Q}{n}\right)^{n-1}Qd\sigma\left(s\right)}_{I(Q)}.\end{aligned}$$

Note that $I(Q)$ does not depend on F. Then deterministic supply is (weakly) optimal. $\square$

In light of Proposition 1.2, once the seller commits to deterministic supply they are no longer guaranteed the revenue implied by robust bids, and equilibrium revenue may discontinuously fall. However, the seller can ensure at least the revenue implied by robust bidding by placing ε probability on unbounded supply, for any $\varepsilon > 0$.

Remark 1. Unlike ex ante transparency, ex post transparency (Theorem 6 of PW) cannot be extended to the uniform-price format. Once the auctioneer announces the realized quantity, Proposition 1.2 implies that any price $p^{\star}$ between R and $v^{-1}(Q;s)$ is feasible in the subsequent bidding equilibrium, and ex post revenue may fall in all cases. Thus in the uniform-price auction ex ante uncertainty may be valuable.

2 The Design of Optimal Pay-as-Bid Auctions

2.1 Equivalence of Reserve Prices and Supply Restriction

Rearranging the equilibrium bid expression in Theorem 3 of PW gives the following:

$$b^i(q) = \int_{\min\left\{nq,\overline{Q}^R\right\}}^{\overline{Q}^R} v\left(\frac{x}{n}\right) dF^{nq,n}(x) + v\left(\frac{\overline{Q}^R}{n}\right)\left(1 - F^{nq,n}\left(\overline{Q}^R\right)\right).$$

This immediately implies the following equivalence between reserve prices and a particular change in supply distribution:

Corollary 1. [Reserve Price as Supply Restriction] *Suppose $v(q;s) = v(q)$ for every quantity q and signal s. For every reserve price R there is a reduction of supply that is revenue equivalent to imposing R.*

Without bidder information all reserve prices can be mimicked by supply decisions, but not all supply decisions can be mimicked by the choice of reserve prices: reserve prices imply atoms in the quantity distribution, thus distributions without atoms cannot be induced by a (binding) reserve price. Notice also that, with concentrated distributions, our results imply that attracting an additional bidder is more profitable then setting the reserve price right.[2]

Because optimal supply is deterministic (Theorem 5 of PW), the arguments underpinning Corollary 1 also imply the following.

Corollary 2. [Optimal Reserve Price] *Suppose $v(q;s) = v(q)$ for every quantity q and signal s. The optimal reserve price R is equal to bidders' marginal value at the optimal deterministic supply: $R \in \max_{R'} R'v^{-1}(R')$.*

When the reserve price R is binding, the equivalence between reserve prices and supply restrictions gives an effective maximum supply of $\overline{Q}^R = nv^{-1}(R)$. At this quantity, parceled over each agent, each agent's bid will equal her marginal value, as at $\overline{Q}$ in the unrestricted case. Since bids fall below values, this bid is weakly above the bid placed at this quantity when there is no reserve price. For quantities below $\overline{Q}^R$ the c.d.f. is unchanged, hence our representation and uniqueness theorems combine to imply that the bids submitted with a reserve price will be higher than without. These effects can be seen in Figure 1.

2.2 Separability of Optimal Supply and Reserve Price

The transparency result (Theorem 5 of PW) substantially simplifies the seller's optimization problem. With reserve price R and deterministic supply Q—recall that optimal supply is

[2] See, e.g., [Bulow and Klemperer, 1996] for a similar analysis of the single-unit case.

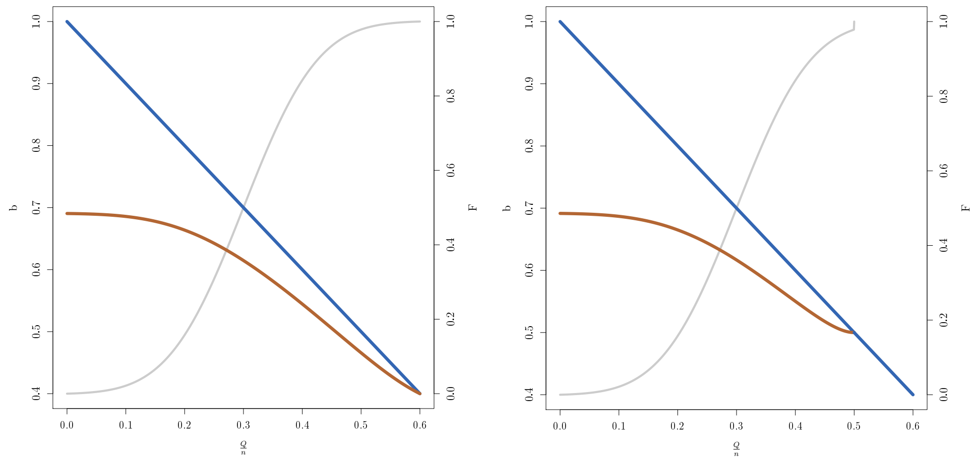

Figure 1: The equilibrium bid function with normal distribution of supply (left), with optimal reserve price (right). The bid for the implicit "maximum quantity" equals the marginal value for this quantity, and the entire bid function shifts up.

deterministic—the seller's revenue is

$$\begin{aligned}\mathbb{E}_s\left[\pi\right] = &\Pr\left(v\left(\frac{Q}{n};s\right) \geq R\right)\mathbb{E}\left[v\left(\frac{Q}{n};s\right)\middle| v\left(\frac{Q}{n};s\right) \geq R\right] Q \\ &+ \Pr\left(v\left(\frac{Q}{n};s\right) < R\right) R\mathbb{E}\left[nv^{-1}\left(R;s\right)\middle| v\left(\frac{Q}{n};s\right) < R\right].\end{aligned}$$

Because in Theorem 3 of PW we show that equilibrium strategies are symmetric, revenue depends on whether the marginal value for the per-capita quantity available, Q/n, is above or below the reserve price R.[3]

We now show that the seller's optimization problem is separable in supply and reserve. To see this, we consider sets $\underline{\mathcal{S}}(Q,R) = \{s\colon v(Q/n;s) < R\}$ and $\overline{\mathcal{S}}(Q,R) = \{s\colon v(Q/n;s) \geq R\}$. When bidders have common signal $s \in \underline{\mathcal{S}}(Q,R)$, their equilibrium bids are constrained by the reserve price and do not depend on the quantity supplied; when bidders have common signal $s \in \overline{\mathcal{S}}(Q,R)$, their equilibrium bids are constrained by the quantity supplied and do not depend on the reserve price. Consider a common signal on the cusp of $\overline{S}(Q,R)$, with $v(Q/n;s) = R$. A slight increase in reserve to $R' > R$ will cause this signal to be reserve-constrained, so that $s \in \underline{\mathcal{S}}(Q,R')$, but will not affect revenue. A similar argument holds for a slight increase in supply to $Q' > Q$. Then, near optimal supply and reserve, marginal changes in $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ do not affect revenue, and these conditions may be ignored.

[3] Both reserve price and quantity restriction play a role in optimizing pay-as-bid auctions, except in the special case of complete information case which lends itself to some simplifications in the design analysis; see Section 2.1 above.

Theorem 1. [Separable Optimization] *Let $R^\star$ be an optimal reserve and $Q^\star$ be the optimal supply in a pay-as-bid auction. Then*

$$R^\star \in \arg\max_R R\mathbb{E}\left[v^{-1}(R;s) \middle| s \in \underline{\mathcal{S}}(Q^\star, R^\star)\right], \text{ and } Q^\star \in \arg\max_Q Q\mathbb{E}\left[v(Q;s) \middle| s \in \overline{\mathcal{S}}(Q^\star, R^\star)\right].$$

In the proof, we are not restricting attention to signals drawn from a subset of $\mathbb{R}$ and marginal values monotonic in signals. In the general case, define the sets to represent the two possibilities for signal realizations: either the market clearing price is the reserve price, or it exceeds the reserve price. Theorem 3 then takes the form of the claim that the optimal reserve price R^* and quantity Q^* satisfy

Proof. Expected revenue can be expressed as a sum over two integrals,

$$\mathbb{E}_s[\pi] = \int_{s\in\underline{\mathcal{S}}(Q,R)} nR\varphi(R;s)\,d\sigma(s) + \int_{s\in\overline{\mathcal{S}}(Q,R)} Qv\left(\frac{1}{n}Q;s\right)d\sigma(s).$$

From this expression, the seller's choice of optimal (deterministic) quantity and reserve price can be found by taking first-order conditions. Assuming for simplicity that $v(q;\cdot)$ is continuous gives[4]

$$\begin{aligned}\frac{\partial\mathbb{E}_s[\pi]}{\partial R} &= \int_{s\in\underline{\mathcal{S}}(Q,R)} n\varphi(R;s) + nR\varphi_R(R;s)\,d\sigma(s)\\ &\quad + \frac{\partial}{\partial R}\sigma\left(\underline{\mathcal{S}}(Q,R)\right)RQ + \frac{\partial}{\partial R}\sigma\left(\overline{\mathcal{S}}(Q,R)\right)QR\\ &= n\int_{s\in\underline{\mathcal{S}}(Q,R)} \frac{\partial}{\partial R}\left[R\varphi(R;s)\right]d\sigma(s).\end{aligned}$$

Similar calculations imply $\partial\mathbb{E}_s[\pi]/\partial Q = \int_{s\in\overline{\mathcal{S}}(Q,R)}(\partial[Qv(Q/n;s)]/\partial Q)d\sigma(s)$. That is, the problem of selecting optimal supply and reserve price is identical to the decoupled problems of maximizing revenue on $s \in \underline{\mathcal{S}}(Q^\star, R^\star)$ by setting a price, and maximizing revenue on $s \in \overline{\mathcal{S}}(Q^\star, R^\star)$ by setting a quantity, then ensuring consistency of the presumed sets $\underline{\mathcal{S}}(Q^\star, R^\star)$ and $\overline{\mathcal{S}}(Q^\star, R^\star)$. □

When signals are drawn from $\mathbb{R}$ and marginal values are increasing in signal, the sets $\underline{\mathcal{S}}(Q, R)$ and $\overline{\mathcal{S}}(Q, R)$ are expressable in terms of a single threshold $\hat{s} \in \mathbb{R}$, and Theorem 1 takes a simple form.

[4]If $v(q;\cdot)$ were not continuous, the derivatives with respect to the bounds of integration still would cancel: any signal realizations "lost" in the first integral are necessarily "gained" by the second, and vice-versa. Since the definitions of $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ imply that for all $s \in (\mathrm{Cl}\underline{\mathcal{S}}(Q,R)) \cap \overline{\mathcal{S}}(Q,R)$, $nR\varphi(R;s) = Qv(Q/n;s)$ the integrand-mass associated with the shifting boundaries is equal in both integrals, hence the terms cancel regardless of the well-behavedness of $v(q;\cdot)$.

Corollary 3. [Separable Optimization] *Suppose that marginal values are increasing in the bidders' common signal s, and that s is drawn from an atomless distribution on $\mathbb{R}$. Let $R^\star$ be an optimal reserve, $Q^\star$ be the optimal supply in a pay-as-bid auction, and* $\hat{s} = \inf\{s \colon v(Q^\star/n; \hat{s}) \geq R^\star\}$. *Then, $\hat{s} \in \mathbb{R}$ and*

$$R^\star \in \arg\max_R R\mathbb{E}\left[v^{-1}(R;s)\middle|s < \hat{s}\right], \quad Q^\star \in \arg\max_Q Q\mathbb{E}\left[v(Q;s)|s \geq \hat{s}\right].$$

We illustrate the value of Theorem 1 and Corollary 3 in an example in Section 2.5.

2.3 Pay-as-Bid vs. Posted Price vs. Cournot Quantity

Consider now two alternate problems, one in which a standard monopolist posts a price, and one in which the monopolist commits to a quantity. In the former problem, the monopolist solves

$$\max_p n\mathbb{E}_s\left[pv^{-1}(p;s)\right].$$

In the latter problem, the monopolist solves

$$\max_q n\mathbb{E}_s\left[qv(q;s)\right].$$

We now compare optimal supply and reserve in the pay-as-bid auction to the optimal monopoly price, and the optimal monopoly quantity, and show that the auctioneer's optimal reserve is below the optimal monopoly price while the auctioneer's optimal supply is above the optimal monopoly quantity. This comparison arises from the ability of the pay-as-bid seller to hedge the two design parameters against one another. When reserve price is the only instrument available, the seller needs to balance the desire to extract surplus from high-value consumers against the desire to not sacrifice too much quantity with a too-high reserve price against low-value consumers; in the pay-as-bid auction the high-value consumers "self-discriminate," since their unique bid function exactly equals their marginal value when the quantity for sale is deterministic. When quantity is the only instrument available the seller is still balancing the same forces, but the presence of a reserve price ensures that he will not sacrifice too much surplus to low-value consumers when he sets the quantity relatively high. When values are sufficiently regular this argument generalizes in a natural way.[5]

[5] The literature on market regulation has considered whether price or quantity is a better instrument for achieving desired outcomes; the perspective taken is generally that of the regulator, rather than of a monopolist. Weitzman [1974] obtains conditions under which price or quantity regulation is preferred under stochastic demand and supply; Roberts and Spence [1976] find that a three-part system involving permits, penalties, and repurchase is preferable to any single-instrument system.

Proposition 4. [Comparison of Pay-as-Bid Seller to Monopolist] *Let quantity-monopoly profits π^Q be given by $\pi^Q(Q,s) = Qv(Q/n;s)$, and let $\hat{Q}(s) \in \arg\max_q \pi^Q(q,s)$; let price-monopoly profits π^R be given by $\pi^R(R;s) = nR\varphi(R;s)$, and let $\hat{R}(s) \in \arg\max_p \pi^R(p;s)$. Let Q^M be optimal quantity-monopoly supply and R^M be optimal price-monopoly reserve against $s \sim \sigma$, and let $Q^{\star PAB}$ and $R^{\star PAB}$ be the optimal deterministic supply and reserve price from the pay-as-bid seller's problem. If $v(q;\cdot)$ is monotonically increasing for all q,$\pi^Q(\cdot;s)$ is strictly concave for all s and $\hat{Q}(\cdot)$ is monotonically increasing, then $Q^M \le Q^{\star PAB}$; if $v(q;\cdot)$ is monotonically increasing for all q, $\pi^R(\cdot;s)$ is strictly concave for all s and $\hat{R}(\cdot)$ is monotonically increasing, then $R^{\star PAB} \le R^M$.*

Proposition 4 is natural in light of the separability of the designer's optimization problem. For a monopolist, increasing a price is typically more desirable when consumers have higher valuations. A supply restriction in effect cuts high-value consumers out of the price optimization problem—provided the price is not too high, their demand will be constrained to available supply—and it is less advantageous to increase prices. Then optimal reserve prices in a supply-constrained pay-as-bid auction will be below optimal monopoly prices. Similar logic, focusing on cutting low-value consumers out of the market, applies in the comparative analysis of market supply.

Proof of Proposition 4. Consider implementing reserve price R; the condition of quantity optimality at $Q^\star(R)$, the optimal quantity given reserve price R, is

$$0 = \int_{s\in\overline{\mathcal{S}}(Q^\star(R),R)} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma(s).$$

Since $\pi(\cdot;s)$ is strictly concave and $\hat{Q}(\cdot)$ is monotonically increasing, for any Q either $\pi^Q_Q(Q;s) < 0$ for all s, or $\pi^Q_Q(Q;s) > 0$ for all s, or there is some $\bar{s}$ such that $\pi^Q_Q(Q;s') \le 0$ for all $s' > \bar{s}$ and $\pi^Q_Q(Q;s') \ge 0$ for all $s' > \bar{s}$. Neither of the first two cases support the optimality condition above, hence there is $\bar{s} \in \overline{\mathcal{S}}(Q^\star(R),R)$ such that $\pi^Q_Q(Q^\star(R);s') \le 0$ for all $s > s'$ and $\pi^Q_Q(Q^\star(R);s') \ge 0$ for all $s < \bar{s}$. Then we have

$$\int_{s\in\overline{\mathcal{S}}(Q^\star(R),R)} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right) \ge \int_{\mathrm{Supp}\sigma} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right).$$

Since $\pi^Q(\cdot;s)$ is strictly concave for all s, whenever $Q < Q^M$, $\pi^Q_Q(Q;s) > \pi^Q_Q(Q^M;s)$. Then if $Q^\star < Q^M$, we have

$$\int_{\mathrm{Supp}\sigma} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right) > \int_{\mathrm{Supp}\sigma} \frac{\partial}{\partial Q}\pi^Q\left(Q^M;s\right)d\sigma\left(s\right).$$

Putting these inequalities together gives

$$0=\int_{s\in\overline{\mathcal{S}}(Q^{\star}(R);R)}\frac{\partial}{\partial Q}\pi^{Q}\left(Q^{\star}\left(R\right);s\right)d\sigma\left(s\right)>\int_{\mathrm{Supp}\sigma}\frac{\partial}{\partial Q}\pi^{Q}\left(Q^{M};s\right)d\sigma\left(s\right)=0.$$

This is a contradiction, hence $Q^M \leq Q^{\star\mathrm{PAB}}$. A similar argument applies to the case of $R^{\star\mathrm{PAB}} \leq R^M$. □

Remark 2. Since the optimal pay-as-bid auction sells a greater quantity at a lower reserve price, it is more efficient (generates greater total surplus) than either standard monopoly problem.

2.4 Comparative Statics: The Effect of Bidder Signal Distribution on Revenue

As an illustration of the separability theorem (Theorem 1), consider the issue whether the seller benefits from a mean-preserving spread of the distribution of bidders' signal.[6]

Proposition 5. [Comparative Statics] *If marginal values are increasing in the bidders' common signal, and are linear and additively separable in quantity and signal, then the seller's revenue in optimally designed pay-as-bid auction is increased by any mean-preserving spread of the distribution of bidders' signals.*

Proof. The assumptions of the proposition allow us to linearly renormalize the signal s so as to make it one-dimensional and represent the marginal revenue as $v(q;s)=s-\rho q$. We conduct the proof under the assumption that the distribution σ has no atoms; because any distribution σ can be approximated via atomless distributions and the seller maximization is continuous with respect to such approximations, imposing this assumption is without loss of generality. Let $\pi(s)$ be the equilibrium revenue associated with signal s and notice that Theorem 3 implies that π is differentiable in s except possibly at the threshold signal $s=\hat{s}$:

For $s<\hat{s}$, we have

$$\frac{d\pi}{ds}=\frac{d}{ds}\left[R^{\star}v^{-1}\left(R^{\star};s\right)\right]=R^{\star}\frac{d}{ds}\left[\frac{1}{\rho}\left(s-R^{\star}\right)\right]=\frac{R^{\star}}{\rho}.$$

For $s>\hat{s}$, we have

$$\frac{d\pi}{ds}=\frac{d}{ds}\left[Q^{\star}v\left(Q^{\star};s\right)\right]=Q^{\star}\frac{d}{ds}\left[s-\rho Q^{\star}\right]=Q^{\star}.$$

[6] We explore this issue further in the example in Section 2.5.

Thus, $\frac{d\pi}{ds}$ is piecewise constant and its value for $s < \hat{s}$ is strictly below its value for $s > \hat{s}$ because $\frac{R^\star}{\rho} < Q^\star$; the latter inequality being satisfied because Theorem 3 gives

$$R^\star = \frac{1}{2}\mathbb{E}\left[s|s < \hat{s}\right],$$
$$Q^\star = \frac{1}{2\rho}\mathbb{E}\left[s|s > \hat{s}\right]$$

(where the strict inequality follows from σ being atomless). Furthermore, at $s = \hat{s}$, $\frac{d\pi}{ds}$ has side derivatives as above. Defining $\frac{d\pi}{ds}|_{s=\hat{s}}$ to be a value weakly between the side derivatives, we find that $\frac{d\pi}{ds}$ that is convex.

Now, consider an alternate signal distribution σ'. As with σ we can assume that this distribution is atomless.[7] The above analysis implies that

$$\mathbb{E}\left[\pi\right] = \int_{\underline{s}}^{\overline{s}} \pi\left(s\right) d\sigma\left(s\right) = \pi\left(\underline{s}\right) + \int_{\underline{s}}^{\overline{s}} \frac{d\pi}{ds}\left(1 - \sigma\left(s\right)\right) ds,$$

where $\underline{s}, \overline{s}$ are such that $\operatorname{Supp}\sigma, \operatorname{Supp}\sigma' \subseteq [\underline{s}, \overline{s}]$; note that the value we set for $\frac{d\pi}{ds}|_{s=\hat{s}}$ doesn't matter because of the assumption that σ is atomless. The optimal revenue under distribution σ' is bounded below by the revenue obtained with the reserve $R^\star$ and quantity $Q^\star$ that are optimal for distribution σ, and thus the difference in optimal revenues is at least

$$\begin{aligned}
\mathbb{E}_{s\sim\sigma'}\left[\pi\left(s\right)\right] - \mathbb{E}_{s\sim\sigma}\left[\pi\left(s\right)\right] &= \int_{\underline{s}}^{\overline{s}} \mu\pi\left(s\right)\left(\sigma\left(s\right) - \sigma'\left(s\right)\right) ds \\
&= \int_{\underline{s}}^{\overline{s}} \frac{R^\star}{\rho}\left(\sigma\left(s\right) - \sigma'\left(s\right)\right) ds + \int_{\hat{s}}^{\overline{s}} \left(Q^\star - \frac{R^\star}{\rho}\right)\left(\sigma\left(s\right) - \sigma'\left(s\right)\right) ds \\
&= \frac{R^\star}{\rho}\left(\mathbb{E}_{s\sim\sigma}\left[s\right] - \mathbb{E}_{s\sim\sigma'}\left[s\right]\right) + \left(Q^\star - \frac{R^\star}{\rho}\right)\left(\int_{\hat{s}}^{\overline{s}} \sigma\left(s\right) - \sigma'\left(s\right) ds\right).
\end{aligned}$$

If σ' is a mean-preserving spread of σ, the left-hand term is (definitionally) zero. Because, as noted above, $Q^\star > R^\star/\rho$, a mean-preserving spread σ' improves revenue if (but not necessarily only if)

$$\int_{\hat{s}}^{\overline{s}} \sigma\left(s\right) - \sigma'\left(s\right) ds > 0.$$

The latter condition is always satisfied when σ' is a mean-preserving spread of σ. □

Remark 3. Proposition 5 remains valid when the maximum feasible supply is below optimal

[7] As later we will assume that σ' is a mean-preserving spread of σ, it is also important that we can approximate such σ and σ' by sequences of atomless distributions σ_k and σ'_k that converge to σ and σ', respectively, and that are such that σ'_k is a mean-preserving spread of σ_k. To find such sequences, we can, for instance, set σ_k to be $\sigma + U_{\frac{1}{k}}$ and σ'_k to be $\sigma' + U_{\frac{1}{k}}$ where $U_{\frac{1}{k}}$ is the uniform distribution on $\left[-\frac{1}{k}, \frac{1}{k}\right]$.

monopoly supply, $Q^{\max} < Q^{\star}$. In this case, the optimal feasible quantity is $Q^{\max}$, and the optimal reserve is $R^{\max} < R^{\star}$. At the threshold signal $\hat{s}^{\max}$, it is the case that $R^{\max} = \hat{s}^{\max} - \rho Q^{\max}$, and $Q^{\max} > R^{\max}/\rho$. All derivations in Proposition 5 remain valid.

2.5 Example

Example 1. Take some constants $\rho, \underline{s}, \overline{s} > 0$, such that $\overline{s} > \underline{s} \geq \rho \overline{Q}/n$ and suppose that s is distributed uniformly on $(\underline{s}, \overline{s})$ and $v(q; s) = s - \rho q$ for some constant $\rho > 0$. Thus, $\varphi(R; s) = (s - R)/\rho$. For every relevant deterministic supply Q and reserve price R is then the unique cut-off $\tau = \tau(Q, R) = R + \rho Q/n$ such that[8]

$$R = v\left(\frac{Q}{n}; \tau\right) = \tau - \rho \frac{Q}{n}.$$

For all $s < \tau(Q, R)$ the seller sells quantity $\varphi(R; s) = n(s - R)/\rho$ at price R; for all $s > \tau(Q, R)$ the seller sells quantity Q at price $v(Q/n; s) = s - \rho Q/n$. Following Corollary 3, the seller's two-part maximization problem is[9]

$$\max_R \mathbb{E}_s \left[n \left(\frac{s - R}{\rho} \right) R \middle| s < \tau\left(Q^{\star}, R^{\star}\right) \right], \text{ and } \max_Q \mathbb{E}_s \left[\left(s - \frac{\rho Q}{n} \right) Q \middle| s > \tau\left(Q^{\star}, R^{\star}\right) \right].$$

This gives two equations,

$$\mathbb{E}_s\left[s | s < \tau\left(Q^{\star}, R^{\star}\right)\right] - 2R^{\star} = 0, \text{ and } n\mathbb{E}_s\left[s | s > \tau\left(Q^{\star}, R^{\star}\right)\right] - 2\rho Q^{\star} = 0.$$

Note that, after derivatives have been taken, we replace Q with $Q^{\star}$. Since $\mathbb{E}_s[s|s < \tau(Q^{\star}, R^{\star})] = \underline{s} + (\tau(Q^{\star}, R^{\star}) - \underline{s})/2$ and $\mathbb{E}_s[s|s > \tau(Q^{\star}, R^{\star})] = \overline{s} + (\tau(Q^{\star}, R^{\star}) - \overline{s})/2$, substituting in yields

$$2R^{\star} = \underline{s} + \frac{1}{2}\left(R^{\star} + \frac{1}{n}\rho Q^{\star} - \underline{s}\right), \text{ and } 2\frac{1}{n}\rho Q^{\star} = \overline{s} + \frac{1}{2}\left(R^{\star} + \frac{1}{n}\rho Q^{\star} - \overline{s}\right).$$

Then

$$R^{\star} = \frac{\overline{s} + 3\underline{s}}{8}, \text{ and } \frac{1}{n}Q^{\star} = \frac{3\overline{s} + \underline{s}}{8\rho}.$$

This immediately implies that $\tau = (\overline{s} + \underline{s})/2$.

[8] Note that since signals are uni-dimensional and values are strictly monotone in signal, the sets $\underline{\mathcal{S}}(Q, R)$ and $\overline{\mathcal{S}}(Q, R)$ are uniquely identified with such a cut-off τ.

[9] Since the uniform distribution is massless, we can ignore the event $s = \tau$. Also, for expositional purposes we constrain attention to cases in which the seller's problem has an interior solution.

Comparison to standard monopolists

The standard monopoly problems are straightforward. The quantity-monopoly problem is

$$\max_Q \mathbb{E}_s \left[Qv\left(\frac{Q}{n}; s\right)\right] = \max_Q Qv\left(\frac{Q}{n}; \mathbb{E}_s[s]\right) = \max_Q \left(\frac{\overline{s}+\underline{s}}{2} - \rho Q\right) Q.$$

Then optimal quantity is $Q^M = (\overline{s}+\underline{s})/(4\rho)$. The price-monopoly problem is

$$\max_R \mathbb{E}_s [nR\varphi(R;s)] \propto \max_R R\varphi(R;\mathbb{E}_s[s]) \propto \max_Q \left(\frac{\overline{s}+\underline{s}}{2} - R\right) R.$$

Then optimal price is $R^M = (\overline{s}+\underline{s})/4$.

$$\max_p n\mathbb{E}_s \left[\frac{1}{\rho}(s-p)\,p\right].$$

In the latter problem, the monopolist solves

$$\max_q \mathbb{E}_s \left[\left(s - \frac{\rho q}{n}\right) q\right].$$

Then $R^M = (\overline{s}+\underline{s})/4 > (\overline{s}+3\underline{s})/8$, and $R^M > R^{\star\text{PAB}}$; further, $Q^M = n(\overline{s}+\underline{s})/4\rho < n(3\overline{s}+\underline{s})/8\rho$, and $Q^M < Q^{\star\text{PAB}}$. As shown in Proposition 4, the optimally designed pay-as-bid auction allocates a higher quantity at a lower (reserve) price than the classical monopolist's problem.

Effect of variance of bidder signal

The cutoff type is $\hat{s} = \frac{\overline{s}+\underline{s}}{2}$ and the expected revenue is $\frac{n}{2\rho}\left(\left(\frac{3\overline{s}+\underline{s}}{8}\right)^2 + \left(\frac{\overline{s}+3\underline{s}}{8}\right)^2\right)$, which we can express in terms of the mean $m = \frac{\overline{s}+\underline{s}}{2}$ and the variance $V = \frac{(\overline{s}-\underline{s})^2}{12}$ of the signal distribution:

$$\text{Expected Revenue} = \frac{n}{2\rho}\left(\frac{m^2}{2} + \frac{3V}{8}\right).$$

Expected revenue is directly proportional to the number of bidders, a somewhat surprising consequence of the linearity of the problem. The expected revenue is also increasing in the mean and variance of the signal distribution (Proposition 5) and decreasing in the steepness ρ of the marginal value function. The monotonicity in the variance of the distribution means that a mean-preserving spread induces gains on high types that outweigh the losses on low types even when the seller doesn't know the types; the is further able to limit the downside by setting the reserve.

2.6 Old calculations

Example 2. The solution is

$$R^\star = \frac{\overline{s} + 3\underline{s}}{8}, \quad Q^\star = \left(\frac{3\overline{s} + \underline{s}}{8\rho}\right) n.$$

This solution gives the optimal deterministic supply of $Q^\star$ and the optimal reserve price of $R^\star$ provided $Q^\star \leq \overline{Q}$.[10] The reserve price is binding because $\overline{s} > \underline{s}$ implies that $R^\star > \frac{7\underline{s} - 3\overline{s}}{8} = \underline{s} - \rho\frac{Q^\star}{n} = v\left(\frac{Q^\star}{n}; \underline{s}\right)$.

The cutoff type is $\hat{s} = \frac{\overline{s}+\underline{s}}{2}$ and the expected revenue is $\frac{n}{2\rho}\left(\left(\frac{3\overline{s}+\underline{s}}{8}\right)^2 + \left(\frac{\overline{s}+3\underline{s}}{8}\right)^2\right)$, which we can express in terms of the mean $m = \frac{\overline{s}+\underline{s}}{2}$ and the variance $V = \frac{(\overline{s}-\underline{s})^2}{12}$ of the signal distribution:

$$\text{Expected Revenue} = \frac{n}{2\rho}\left(\frac{m^2}{2} + \frac{3V}{8}\right).$$

The expected revenue is proportional to the number of bidders, a somewhat surprising consequence of the linearity of the problem. The expected revenue is also increasing in the mean and variance of the signal distribution and decreasing in the steepness ρ of the marginal value function. The monotonicity in the variance of the distribution means that a mean-preserving spread induces gains on high types that outweigh the losses on low types even when the seller doesn't know the types; the seller benefits from the upside while being able to limit the downside by setting the reserve price.

The separability of Theorem 3 allows us to compare optimally designed pay-as-bid reserve and supply to choices of a seller who sets the price (without optimizing over supply) and to the seller sets the supply allowing the price to be determined by Cournot-like market forces. In the context of the above example, the optimal price is half of the mean valuation for the initial unit, $p^{\text{MONOP}} = \frac{m}{2}$, and the optimal supply is $q^{\text{MONOP}} = \frac{m}{2\rho}n$ (that is the mean type utility on the optimal per-bidder supply is half of the utility on the initial unit), and hence

$$p^{\text{MONOP}} > R^\star \ \text{ and } \ q^{\text{MONOP}} < Q^\star.$$

That is, the optimally designed pay-as-bid auction allocates a higher quantity at a lower (reserve) price than the classical monopolist's problem. This feature arises from the ability of the pay-as-bid seller to hedge the two design parameters against one another. In Supplementary Appendix 2.3, we establish these comparisons more generally, thus showing how Theorem 3 contributes to the literature on whether price or quantity is a better instrument

[10] If $Q^\star > \overline{Q}$ then $Q = \overline{Q}$ is the optimal supply and the optimal reserve price $R = \frac{\rho\overline{Q}}{3n} + \frac{\underline{s}}{3}$ is given by the first order condition.

for achieving desired market outcomes.[11]

The separability of the pay-as-bid designer's problem shown in Theorem 3 sharply contrasts with problem faced by a designer of a uniform-price auction, which we study in Section SECTION: AUCTION DESIGN GAME. In the uniform-price auction equilibrium bids are not (in general) unique, the strict monotonicity of bid in value cannot be assured, and it is not necessarily the case that uniform-price auctions can be optimized in a separable manner.

2.6.1 Details on the optimal supply and reserve with linear demand (Example 1)

Assuming that Q and R are both binding, which we will subsequently verify, the monopolist's problem is[12]

$$\max_{Q,R} \int_{\underline{s}}^{\tau} \frac{n}{\rho}\left(s-R\right) R ds + \int_{\tau}^{\overline{s}} Q\left(s - \frac{\rho Q}{n}\right) ds.$$

Here, $\tau = R + \rho Q/n$. The first-order conditions with respect to Q and R are

$$\begin{aligned}
\frac{\partial}{\partial Q} &: \; 0 = \left[\frac{n}{\rho}\left(\tau - R\right) R\right] \frac{\partial \tau}{\partial Q} - \left[Q\left(\tau - \frac{\rho Q}{n}\right)\right] \frac{\partial \tau}{\partial Q} + \int_{\tau}^{\overline{s}} s - \frac{2\rho Q}{n} ds, \\
\frac{\partial}{\partial R} &: \; 0 = \int_{\underline{s}}^{\tau} \frac{n}{\rho}\left(s - 2R\right) ds + \left[\frac{n}{\rho}\left(\tau - R\right) R\right] \frac{\partial \tau}{\partial R} - \left[Q\left(\tau - \frac{\rho Q}{n}\right)\right] \frac{\partial \tau}{\partial R}.
\end{aligned}$$

Note that $\tau - R = \rho Q/n$ and $\tau - \rho Q/n = R$; then the $\partial\tau/\partial\cdot$ terms additively cancel, leaving

$$\int_{\tau}^{\overline{s}} s - \frac{2\rho Q}{n} ds = 0, \quad \int_{\underline{s}}^{\tau} \frac{n}{\rho}\left(s - 2R\right) ds = 0.$$

In particular, after cancelation the remaining terms are as given in Theorem 3.

Solving the optimality condition associated with $Q^{\star}$ gives

$$\frac{1}{2}\left(\overline{s}^2 - \tau^2\right) - \frac{2\rho Q}{n}\left(\overline{s} - \tau\right) = 0.$$

At an internal solution, $\overline{s} > \tau$, so this expression becomes

$$\frac{1}{2}\left(\overline{s} + \tau\right) - \frac{2\rho Q}{n} = 0.$$

[11] Weitzman [1974] obtains conditions under which price or quantity regulation is preferred under stochastic demand and supply; Roberts and Spence [1976] find that a three-part system involving permits, penalties, and repurchase is preferable to any single-instrument system.

[12] Because the signal distribution is uniform, we ignore the constant of proportionality $1/(\overline{s} - \underline{s})$.

Substituting in for $\tau = R + \rho Q/n$ leaves the expression

$$\frac{1}{2}\left(\overline{s} + R + \frac{\rho Q}{n}\right) - \frac{2\rho Q}{n} = 0 \quad \implies \quad \frac{3\rho Q}{n} = \overline{s} + R.$$

Solving the optimality condition associated with $R^\star$ gives (removing the constant n/ρ)

$$\frac{1}{2}\left(\tau^2 - \underline{s}^2\right) - 2R\left(\tau - \underline{s}\right) = 0.$$

At an internal solution, $\underline{s} < \tau$, so this expression becomes

$$\frac{1}{2}\left(\tau + \underline{s}\right) - 2R = 0.$$

Substituting in for $\tau = R + \rho Q/n$ leaves the expression

$$\frac{1}{2}\left(R + \frac{\rho Q}{n} - \underline{s}\right) - 2R = 0 \quad \implies \quad 3R = \underline{s} + \frac{\rho Q}{n}.$$

Together these equations yield the linear system

$$\begin{aligned} \frac{3\rho Q}{n} =& \overline{s} + R, \\ 3R =& \underline{s} + \frac{\rho Q}{n}. \end{aligned}$$

It is straightforward to see that the solution is

$$Q^\star = \left(\frac{3\overline{s} + \underline{s}}{8\rho}\right) n, \quad R^\star = \frac{\overline{s} + 3\underline{s}}{8}.$$

The signal transition threshold at the optimum is $\tau(Q^\star, R^\star) = (\overline{s} + 3\underline{s})/8 + (3\overline{s} + \underline{s})/8 = (\overline{s}+\underline{s})/2$; then at the optimum both the maximum quantity and the reserve price are binding, as assumed.

The standard monopoly problems are straightforward. The quantity-monopoly problem is

$$\max_Q \mathbb{E}_s \left[Qv\left(\frac{Q}{n}; s\right)\right] = \max_Q Qv\left(\frac{Q}{n}; \mathbb{E}_s\left[s\right]\right) = \max_Q \left(\frac{\overline{s} + \underline{s}}{2} - \rho Q\right) Q.$$

Then optimal quantity is $Q^M = (\overline{s} + \underline{s})/(4\rho)$. The price-monopoly problem is

$$\max_R \mathbb{E}_s \left[nR\varphi\left(R; s\right)\right] \propto \max_R R\varphi\left(R; \mathbb{E}_s\left[s\right]\right) \propto \max_Q \left(\frac{\overline{s} + \underline{s}}{2} - R\right) R.$$

Then optimal price is $R^M = (\overline{s} + \underline{s})/4$.

$$\max_p n\mathbb{E}_s \left[\frac{1}{\rho} (s - p) \, p \right].$$

In the latter problem, the monopolist solves

$$\max_q \mathbb{E}_s \left[\left(s - \frac{\rho q}{n} \right) q \right].$$

Then $p^{\text{MONOP}} = (\overline{s} + \underline{s})/4$. Then $q^{\text{MONOP}} = n(\overline{s} + \underline{s})/4\rho$. Comparing the monopolist's problems to the pay-as-bid seller's problem, we can see that $p^{\text{MONOP}} > R^\star$ and $q^{\text{MONOP}} < Q^\star$: that is, the optimally designed pay-as-bid auction allocates a higher quantity at a lower (reserve) price than the classical monopolist's problem.

This comparison turns out to be robust.

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