Supplementary Note for Pycia and Woodward [2023b]

Marek Pycia* and Kyle Woodward†

August 2023

In this note we provide supplementary results and applications for Pycia and Woodward [2023b], henceforth, "PW". We omit the model, notation, and assumptions, and refer the interested reader to Section 3 of PW.

1 Comparative Statics in Pay-as-Bid Auctions

The bid representation in PW Theorem 3 implies that supply concentration leads to flat bids and low margins on bids near the per-capita concentrated quantity. We say that a distribution is δ-concentrated near quantity Q^* if $1-\delta$ of the mass of supply is within δ of quantity Q^*.

Corollary 1. **[Low Margins]** *For any $\varepsilon > 0$ and quantity $Q^* \leq \overline{Q}^R$ there exists $\delta > 0$ such that, if supply is δ-concentrated near Q^*, then each bidder's equilibrium margin $v\left(\frac{1}{n}Q^* - \delta\right) - b\left(\frac{1}{n}Q^* - \delta\right)$ on the $\frac{1}{n}Q^* - \delta$ unit is lower than ε.*

This corollary complements PW Theorem 1, which establishes that bidders obtain zero margin at the maximum quantity.

2 Equilibrium in Uniform-Price Auctions

2.1 Robust Equilibrium Selection

As noted in PW, the uniform-price auction typically admits multiple equilibria. An exception is described by Klemperer and Meyer [1989], who show that equilibrium bidding strategies are unique when the support of supply is sufficiently large. For a similar argument to hold in the context of PW, it is sufficient that the support of supply is $[0, \overline{Q}(R)]$, where $\overline{Q}(R) \geq$

*University of Zurich; marek.pycia@econ.uzh.ch
†University of North Carolina at Chapel Hill; kyle.woodward@unc.edu

$\sup_s nv^{-1}(R; s)$. Because any supply distribution F can be nearly costlessly transformed to a distribution with unbounded supply—supply can follow distribution F with probability $1-\varepsilon$ and an unbounded, convex distribution with probability ε—we model large supply by assuming that supply is unbounded and convex. We define robust uniform-price bids to be those corresponding to an unbounded distribution of supply. Since equilibrium in the uniform-price auction is ex post, these bids are equilibria given any other distribution of supply and the same reserve price.

Robust uniform-price bids are a natural limit of slight uncertainty about aggregate supply. We define a bid profile to be robust to uncertainty if small changes in the distribution of quantity result in small changes to equilibrium bids.

Definition 1. [Robust Bids] Given supply distribution F and reserve price R, a bid profile $(b^i)_{i=1}^n$ is *robust to uncertainty* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any supply distribution $\tilde{F}$ with $\sup_{Q\in\mathbb{R}} |F(Q) - \tilde{F}(Q)| < \delta$, all equilibrium bid profiles $(\tilde{b}^i)_{i=1}^n$ are such that $\sup_{s,q\in[0,\tilde{Q}^R(s)]} |b^i(q;s) - \tilde{b}^i(q;s)| < \varepsilon$ for all bidders i, where[1]

$$\tilde{Q}^R(s) = \min\left\{\max \mathrm{Supp}_F \frac{1}{n}Q,\ \max \mathrm{Supp}_{\tilde{F}} \frac{1}{n}Q,\ v^{-1}(R;s)\right\}.$$

Proposition 1. [Bids Robust to Uncertainty] *The unique pay-as-bid equilibrium bid profile is robust to uncertainty. The unique uniform-price equilibrium bid profile that is robust to uncertainty is given by:*

$$b(q;s) = \left(\frac{q}{\hat{Q}^R(s)}\right)^{n-1} R + (n-1)\int_q^{\hat{Q}^R(s)} \left(\frac{q}{x}\right)^{n-1} \frac{v(x;s)}{x}dx, \tag{1}$$

where $\hat{Q}^R(s) = v^{-1}(R;s)$.

We refer to the above uniform-price bid profile as *robust uniform-price bids.*

Proof. Equilibrium bids in the pay-as-bid auction are robust to uncertainty because the bid form given in PW Theorem 3 is continuous, with respect to the supremum norm, in supply distribution. With regard to the uniform price auction, note that given any supply distribution F there is an unbounded supply distribution $\tilde{F}$ with $\sup_{Q\in\mathbb{R}} |F(Q) - \tilde{F}(Q)| < \varepsilon$. With unbounded supply, robust uniform-price bids are the unique bidding equilibrium in

[1]Note the distinction between $Q^R(Q,s)$ and $\tilde{Q}^R(s)$: the former is the minimum of Q and aggregate demand at price R when bidders have common signal s. The latter is the minimum upper bound of the supports of the distributions F and $\tilde{F}$, or aggregate demand at R when bidders have common signal s, and is defined below.

the uniform-price auction [Klemperer and Meyer, 1989]. It follows that for a bid profile to be robust given supply distribution F and reserve price R, it must be such that $\sup_{s,q\in[0,\hat{Q}(s)]} |b^i(q;s) - \tilde{b}^i(q;s)| < \delta$ for any $\delta > 0$ and all bidders i, where $\tilde{b}^i(\cdot;s)$ is the robust uniform-price bid when bidder type is s. Then $(b^i)_{i=1}^n$ is a robust uniform-price bid profile. $\square$

Robust uniform-price bids are continuous, differentiable, strictly below marginal values for all $q \in (0, \hat{Q}(s))$, and equal to marginal values for $q \in \{0, \hat{Q}(s)\}$. No matter which auction format is employed, optimal supply $Q^\star > 0$. In the pay-as-bid design game the optimal deterministic quantity must be binding for some bidder types, $Q^{\star\text{PAB}} < \sup_s v^{-1}(R;s)$, provided the value space is rich (as in PW Section 6.1). Since robust uniform-price bids are strictly below value on $(0, Q^{\star\text{PAB}}/n]$ for all types s such that $Q^{\star\text{PAB}} < v^{-1}(R;s)$, the pay-as-bid auction generates strictly greater revenue than the uniform-price auction with robust bidding.

Proposition 2. [Strict Dominance of Pay-as-Bid Revenue] *The pay-as-bid design game generates strictly greater revenue than the unique equilibrium of the uniform-price design game in robust bids.*

In light of Proposition 2 and our main text analysis, combining seller-optimal equilibrium bids on-path and robust bids as off-path threats allows the construction of a variety of PBEs: if the auctioneer implements are particular supply distribution and reserve price, bidders will play the (conditional) seller-optimal equilibrium, and otherwise bidders will play the unique equilibrium in robust uniform-price bids. This implies that any not-too-suboptimal supply distributions and reserve prices can be implemented in a perfect Bayesian equilibrium, as shown in PW Theorem 12 of PW.

2.2 Price Range in Uniform Price

Robust uniform-price bids provide a natural off-path equilibrium to ensure the selection of a bidder-preferred parameterization of the uniform-price auction format (or any other parameterization consistent with PW Theorem 12). Although robust bidding equilibria remain equilibria when supply is deterministic, restricting attention to deterministic supply allows for a significant range of market clearing prices: any market clearing price between the reserve price R and the marginal value for the per-capita quantity $v(Q/n;s)$ can obtain in equilibrium.

Proposition. [Range of Prices in Uniform-Price Design Game] *Let $p^\star(s)$ denote the market-clearing price in an equilibrium of the uniform-price design game with deterministic* *supply Q. Then for all signals s, $p^\star(s) \in [R, \max\{R, v^{-1}(Q/n; s\}]$. Furthermore, for any $p^\star$ such that $p^\star(s) \in [R, \max\{R, v^{-1}(Q/n; s)\}]$ for all s, there is an equilibrium of the uniform-price design game with market-clearing price $p^\star$.*

Proof. To prove the first claim note that $p^\star(s) \geq R$ by definition. If $p^\star(s) > v^{-1}(Q/n; s) > R$, some agent is allocated q_i such that $v(q_i; s) < p^\star(s)$. If she bids $b' = v(\cdot; s)$ instead, she is awarded all units she values above $p^\star(s)$, and possibly more, at a price no greater than $p^\star(s)$. Then she obtains a positive margin on all units received, and possibly decreases her payment; this deviation is profitable.

To prove the second claim, let $p : \operatorname{Supp} s \to [R, \max\{R, v^{-1}(Q/n; s)\}]$ map bidder signals to putative market-clearing prices, and define bids by

$$b\left(q; s, p\right) = \begin{cases} v\left(0; s\right) & \text{if } q < \frac{1}{n}Q, \\ p\left(s\right) & \text{otherwise.} \end{cases}$$

Then downward deviations yield zero quantity, and upward deviations increase the market price without increasing the allocation. Then $(b)_{i=1}^n$ is an equilibrium that yields market clearing price $p(s)$. □

2.3 Transparency in Uniform-Price Auctions

Our main results show that in the pay-as-bid auction, optimal supply is transparent (PW Theorem 5). These results cannot be directly extended to the uniform-price format, because equilibrium multiplicity allows bidders to select qualitatively different equilibria given different parameterizations of the auction. We show now that when the possibility of equilibrium selection is removed, transparency applies as well to the uniform-price format.

Proposition 3. [Transparency in Uniform-Price Auction] *Suppose that given any supply distribution F and reserve price R, bidders play the equilibrium in robust uniform-price bids. Then optimal supply is deterministic.*

Proof. Applying Proposition 1, we have

$$\begin{aligned}\mathbb{E}\left[\pi^{\mathrm{UP}}\right] &= \mathbb{E}_Q \mathbb{E}_s\left[Qp\left(Q; s\right)\right] \\ &= \mathbb{E}_Q \underbrace{\int_{\operatorname{Supp}\sigma} \left[\frac{R}{\hat{Q}\left(s\right)^{n-1}} + (n-1)\int_Q^{\hat{Q}^R(s)} \frac{v\left(x; s\right)}{x^n} dx\right] \left(\frac{Q}{n}\right)^{n-1} Q d\sigma\left(s\right)}_{I(Q)}.\end{aligned}$$

Note that $I(Q)$ does not depend on F. Then deterministic supply is (weakly) optimal. □

In light of Proposition 2.2, once the seller commits to deterministic supply they are no longer guaranteed the revenue implied by robust bids, and equilibrium revenue may discontinuously fall. However, the seller can ensure almost at least the revenue implied by robust bidding by placing ε probability on unbounded supply, for any $\varepsilon > 0$.

Remark 1. Unlike ex ante transparency, ex post transparency (PW Theorem 6) cannot be extended to the uniform-price format. Once the auctioneer announces the realized quantity, Proposition 2.2 implies that any price $p^{\star}$ between R and $\hat{v}^{-1}(Q;s)$ is feasible in the subsequent bidding equilibrium, and ex post revenue may fall in all cases. Thus in the uniform-price auction ex ante uncertainty may be valuable.

3 The Design of Optimal Pay-as-Bid Auctions

3.1 Equivalence of Reserve Prices and Supply Restriction

Rearranging the equilibrium bid expression in PW Theorem 3 gives:

$$b^{i}\left(q\right)=\int_{\min\left\{nq,\overline{Q}^{R}\right\}}^{\overline{Q}^{R}}v\left(\frac{x}{n}\right)dF^{nq,n}\left(x\right)+v\left(\frac{\overline{Q}^{R}}{n}\right)\left(1-F^{nq,n}\left(\overline{Q}^{R}\right)\right).$$

This immediately implies the following equivalence between reserve prices and a particular change in supply distribution:

Corollary 2. [Reserve Price as Supply Restriction] *Suppose $v\left(q;s\right)=v\left(q\right)$ for every quantity q and signal s. For every reserve price R there is a reduction of supply that is revenue equivalent to imposing R.*

Without bidders being more informed than the seller all reserve prices can be mimicked by supply decisions, but not all supply decisions can be mimicked by the choice of reserve prices: reserve prices imply atoms in the quantity distribution, thus distributions without atoms cannot be induced by a (binding) reserve price. In particular, when bidders are no more informed than the seller and the seller sets supply optimally, our results imply that attracting an additional bidder is more profitable than setting the reserve price right.[2]

Because optimal supply is deterministic (PW Theorem 5), the arguments underpinning Corollary 2 also imply the following.

Corollary 3. [Optimal Reserve Price] *Suppose $v\left(q;s\right)=v\left(q\right)$ for every quantity q and signal s. The optimal reserve price R is equal to bidders' marginal value at the optimal deterministic supply: $R\in\max_{R'}R'v^{-1}(R')$.*

[2]See, e.g., [Bulow and Klemperer, 1996] for a similar analysis of the single-unit case.

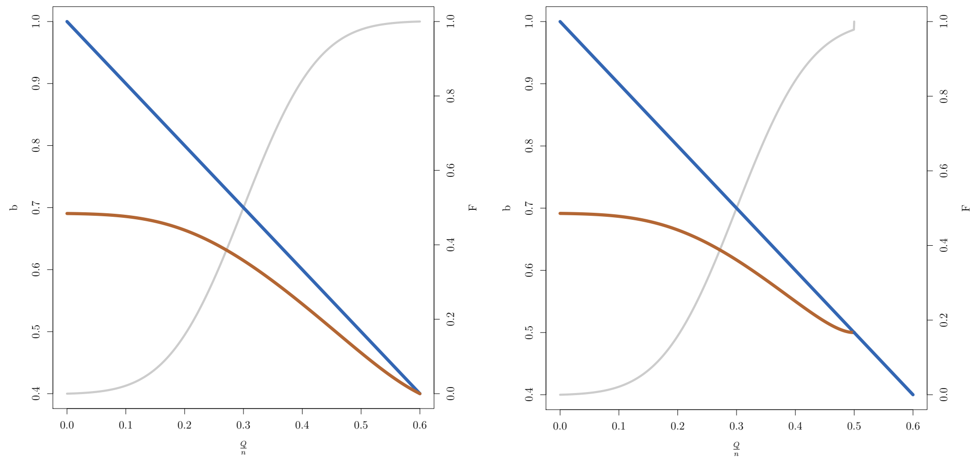

Figure 1: The equilibrium bid function with normal distribution of supply (left), with optimal reserve price (right). The bid for the implicit "maximum quantity" equals the marginal value for this quantity, and the entire bid function shifts up.

When the reserve price R is binding, the equivalence between reserve prices and supply restrictions gives an effective maximum supply of $\overline{Q}^R = nv^{-1}(R)$. At this quantity, parceled over each agent, each agent's bid will equal her marginal value, as at $\overline{Q}$ in the unrestricted case. Since bids fall below values, this bid is weakly above the bid placed at this quantity when there is no reserve price. For quantities below $\overline{Q}^R$ the c.d.f. is unchanged, hence PW's representation and uniqueness theorems (PW Theorems 3 and 4, respectively) combine to imply that the bids submitted with a reserve price will be higher than without. These effects can be seen in Figure 1.

3.2 Separability of Optimal Supply and Reserve Price

The transparency result (PW Theorem 5) substantially simplifies the seller's optimization problem. With reserve price R and deterministic supply Q—recall that optimal supply is deterministic—the seller's revenue is

$$\begin{aligned}\mathbb{E}_s\left[\pi\right] = &\Pr\left(v\left(\frac{Q}{n};s\right) \geq R\right)\mathbb{E}\left[v\left(\frac{Q}{n};s\right)\middle| v\left(\frac{Q}{n};s\right) \geq R\right]Q \\ &+ \Pr\left(v\left(\frac{Q}{n};s\right) < R\right)R\mathbb{E}\left[nv^{-1}\left(R;s\right)\middle| v\left(\frac{Q}{n};s\right) < R\right].\end{aligned}$$

Because in PW Theorem 3 we show that equilibrium strategies are symmetric, revenue depends on whether the marginal value for the per-capita quantity available, Q/n, is above or below the reserve price R.[3]

We now show that the seller's optimization problem is separable in supply and reserve. To see this, we consider sets $\underline{\mathcal{S}}(Q,R) = \{s\colon v(Q/n;s) < R\}$ and $\overline{\mathcal{S}}(Q,R) = \{s\colon v(Q/n;s) \geq R\}$. When bidders have common signal $s \in \underline{\mathcal{S}}(Q,R)$, their equilibrium bids are constrained by the reserve price and do not depend on the quantity supplied; when bidders have common signal $s \in \overline{\mathcal{S}}(Q,R)$, their equilibrium bids are constrained by the quantity supplied and do not depend on the reserve price. Consider a common signal on the cusp of $\overline{S}(Q,R)$, with $v(Q/n;s) = R$. A slight increase in reserve to $R' > R$ will cause this signal to be reserve-constrained, so that $s \in \underline{\mathcal{S}}(Q,R')$, but will not affect revenue. A similar argument holds for a slight increase in supply to $Q' > Q$. Then, near optimal supply and reserve, marginal changes in $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ do not affect revenue, and these conditions may be ignored.

Theorem 1. [Separable Optimization] *Let $R^\star$ be an optimal reserve and $Q^\star$ be the optimal supply in a pay-as-bid auction. Then*

$$R^\star \in \arg\max_R R\mathbb{E}\left[v^{-1}\left(R;s\right)\middle| s \in \underline{\mathcal{S}}\left(Q^\star,R^\star\right)\right], \text{ and } Q^\star \in \arg\max_Q Q\mathbb{E}\left[v\left(Q;s\right)\middle| s \in \overline{\mathcal{S}}\left(Q^\star,R^\star\right)\right].$$

In the proof, we are not restricting attention to signals drawn from a subset of $\mathbb{R}$ and marginal values monotonic in signals. In the general case, define the sets $\underline{\mathcal{S}}(Q,R)$ and $\overline{\mathcal{S}}(Q,R)$ to represent the two possibilities for signal realizations: either the market clearing price is the reserve price, or it exceeds the reserve price. Theorem 4 then takes the form of the claim that the optimal reserve price R^* and quantity Q^* satisfy

Proof. Expected revenue can be expressed as a sum over two integrals,

$$\mathbb{E}_s\left[\pi\right] = \int_{s\in\underline{\mathcal{S}}(Q,R)} nR\varphi\left(R;s\right)d\sigma\left(s\right) + \int_{s\in\overline{\mathcal{S}}(Q,R)} Qv\left(\frac{1}{n}Q;s\right)d\sigma\left(s\right).$$

From this expression, the seller's choice of optimal (deterministic) quantity and reserve price can be found by taking first-order conditions. Assuming for simplicity that $v(q;\cdot)$ is

[3]Both reserve price and quantity restriction play a role in optimizing pay-as-bid auctions, except in the special case of complete information case which lends itself to some simplifications in the design analysis; see Section 3.1 above.

continuous gives[4]

$$\begin{aligned}\frac{\partial \mathbb{E}_s[\pi]}{\partial R} &= \int_{s\in\underline{\mathcal{S}}(Q,R)} n\varphi(R;s) + nR\varphi_R(R;s)\,d\sigma(s) \\ &\quad + \frac{\partial}{\partial R}\sigma\left(\underline{\mathcal{S}}(Q,R)\right)RQ + \frac{\partial}{\partial R}\sigma\left(\overline{\mathcal{S}}(Q,R)\right)QR \\ &= n\int_{s\in\underline{\mathcal{S}}(Q,R)} \frac{\partial}{\partial R}\left[R\varphi(R;s)\right]d\sigma(s).\end{aligned}$$

Similar calculations imply $\partial\mathbb{E}_s[\pi]/\partial Q = \int_{s\in\overline{\mathcal{S}}(Q,R)}(\partial[Qv(Q/n;s)]/\partial Q)d\sigma(s)$. That is, the problem of selecting optimal supply and reserve price is identical to the decoupled problems of maximizing revenue on $s\in\underline{\mathcal{S}}(Q^\star,R^\star)$ by setting a price, and maximizing revenue on $s\in\overline{\mathcal{S}}(Q^\star,R^\star)$ by setting a quantity, then ensuring consistency of the presumed sets $\underline{\mathcal{S}}(Q^\star,R^\star)$ and $\overline{\mathcal{S}}(Q^\star,R^\star)$. □

When signals are drawn from $\mathbb{R}$ and marginal values are increasing in signal, the sets $\underline{\mathcal{S}}(Q,R)$ and $\overline{\mathcal{S}}(Q,R)$ are expressable in terms of a single threshold $\hat{s}\in\mathbb{R}$, and Theorem 1 takes a simple form.

Corollary 4. [Separable Optimization] *Suppose that marginal values are increasing in the bidders' common signal s, and that s is drawn from an atomless distribution on $\mathbb{R}$. Let $R^\star$ be an optimal reserve, $Q^\star$ be the optimal supply in a pay-as-bid auction, and $\hat{s} = \inf\{s\colon v(Q^\star/n;\hat{s})\geq R^\star\}$. Then, $\hat{s}\in\mathbb{R}$ and*

$$R^\star\in\arg\max_R R\mathbb{E}\left[v^{-1}(R;s)\middle|s<\hat{s}\right],\quad Q^\star\in\arg\max_Q Q\mathbb{E}\left[v(Q;s)|s\geq\hat{s}\right].$$

We illustrate the value of Theorem 1 and Corollary 4 in an example in Section 3.5.

3.3 Pay-as-Bid vs. Posted Price vs. Cournot Quantity

Consider now two alternate problems, one in which a standard monopolist posts a price, and one in which the monopolist commits to a quantity. In the former problem, the monopolist solves

$$\max_p n\mathbb{E}_s\left[pv^{-1}(p;s)\right].$$

[4]If $v(q;\cdot)$ were not continuous, the derivatives with respect to the bounds of integration still would cancel: any signal realizations "lost" in the first integral are necessarily "gained" by the second, and vice-versa. Since the definitions of $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ imply that for all $s\in(\mathrm{Cl}\underline{\mathcal{S}}(Q,R))\cap\overline{\mathcal{S}}(Q,R)$, $nR\varphi(R;s)=Qv(Q/n;s)$ the integrand-mass associated with the shifting boundaries is equal in both integrals, hence the terms cancel regardless of the well-behavedness of $v(q;\cdot)$.

In the latter problem, the monopolist solves

$$\max_q n\mathbb{E}_s\left[qv\left(q;s\right)\right].$$

We now compare optimal supply and reserve in the pay-as-bid auction to the optimal monopoly price, and the optimal monopoly quantity, and show that the auctioneer's optimal reserve is below the optimal monopoly price while the auctioneer's optimal supply is above the optimal monopoly quantity. This comparison arises from the ability of the pay-as-bid seller to hedge the two design parameters against one another. When reserve price is the only instrument available, the seller needs to balance the desire to extract surplus from high-value consumers against the desire to not sacrifice too much quantity with a too-high reserve price against low-value consumers; in the pay-as-bid auction the high-value consumers "self-discriminate," since their unique bid function exactly equals their marginal value when the quantity for sale is deterministic. When quantity is the only instrument available the seller is still balancing the same forces, but the presence of a reserve price ensures that he will not sacrifice too much surplus to low-value consumers when he sets the quantity relatively high. When values are sufficiently regular this argument generalizes in a natural way.[5]

Proposition 4. [Comparison of Pay-as-Bid Seller to Monopolist] *Let quantity-monopoly profits π^Q be given by $\pi^Q(Q,s)=Qv(Q/n;s)$, and let $\hat{Q}(s)\in\arg\max_q\pi^Q(q,s)$; let price-monopoly profits π^R be given by $\pi^R(R;s)=nR\varphi(R;s)$, and let $\hat{R}(s)\in\arg\max_p\pi^R(p;s)$. Let Q^M be optimal quantity-monopoly supply and R^M be optimal price-monopoly reserve against $s\sim\sigma$, and let $Q^{\star PAB}$ and $R^{\star PAB}$ be the optimal deterministic supply and reserve price from the pay-as-bid seller's problem. If $v(q;\cdot)$ is monotonically increasing for all q,$\pi^Q(\cdot;s)$ is strictly concave for all s and $\hat{Q}(\cdot)$ is monotonically increasing, then $Q^M\leq Q^{\star PAB}$; if $v(q;\cdot)$ is monotonically increasing for all q, $\pi^R(\cdot;s)$ is strictly concave for all s and $\hat{R}(\cdot)$ is monotonically increasing, then $R^{\star PAB}\leq R^M$.*

Proposition 4 is natural in light of the separability of the designer's optimization problem. For a monopolist, increasing a price is typically more desirable when consumers have higher valuations. A supply restriction in effect cuts high-value consumers out of the price optimization problem—provided the price is not too high, their demand will be constrained to available supply—and it is less advantageous to increase prices. Then optimal reserve prices in a supply-constrained pay-as-bid auction will be below optimal monopoly prices. Similar

[5]The literature on market regulation has considered whether price or quantity is a better instrument for achieving desired outcomes; the perspective taken is generally that of the regulator, rather than of a monopolist. Weitzman [1974] obtains conditions under which price or quantity regulation is preferred under stochastic demand and supply; Roberts and Spence [1976] find that a three-part system involving permits, penalties, and repurchase is preferable to any single-instrument system.

logic, focusing on cutting low-value consumers out of the market, applies in the comparative analysis of market supply.

Proof of Proposition 4. Consider implementing reserve price R; the condition of quantity optimality at $Q^{\star}(R)$, the optimal quantity given reserve price R, is

$$0=\int_{s\in\overline{\mathcal{S}}(Q^{\star}(R),R)}\frac{\partial}{\partial Q}\pi^{\delta_Q}\left(Q^{\star}\left(R\right);s\right)d\sigma(s).$$

Since $\pi(\cdot;s)$ is strictly concave and $\hat{Q}(\cdot)$ is monotonically increasing, for any Q either $\pi_Q^{\delta_Q}(Q;s)<0$ for all s, or $\pi_Q^{\delta_Q}(Q;s)>0$ for all s, or there is some $\bar{s}$ such that $\pi_Q^{\delta_Q}(Q;s')\leq 0$ for all $s'>\bar{s}$ and $\pi_Q^{\delta_Q}(Q;s')\geq 0$ for all $s'>\bar{s}$. Neither of the first two cases support the optimality condition above, hence there is $\bar{s}\in\overline{\mathcal{S}}(Q^{\star}(R),R)$ such that $\pi_Q^{\delta_Q}(Q^{\star}(R);s')\leq 0$ for all $s>s'$ and $\pi_Q^{\delta_Q}(Q^{\star}(R);s')\geq 0$ for all $s<\bar{s}$. Then we have

$$\int_{s\in\overline{\mathcal{S}}(Q^{\star}(R),R)}\frac{\partial}{\partial Q}\pi^{\delta_Q}\left(Q^{\star}\left(R\right);s\right)d\sigma\left(s\right)\geq\int_{\mathrm{Supp}\sigma}\frac{\partial}{\partial Q}\pi^{\delta_Q}\left(Q^{\star}\left(R\right);s\right)d\sigma\left(s\right).$$

Since $\pi^{Q}(\cdot;s)$ is strictly concave for all s, whenever $Q<Q^M$, $\pi_Q^Q(Q;s)>\pi_Q^Q(Q^M;s)$. Then if $Q^{\star}<Q^M$, we have

$$\int_{\mathrm{Supp}\sigma}\frac{\partial}{\partial Q}\pi^{\delta_Q}\left(Q^{\star}\left(R\right);s\right)d\sigma\left(s\right)>\int_{\mathrm{Supp}\sigma}\frac{\partial}{\partial Q}\pi^{\delta_Q}\left(Q^{M};s\right)d\sigma\left(s\right).$$

Putting these inequalities together gives

$$0=\int_{s\in\overline{\mathcal{S}}(Q^{\star}(R);R)}\frac{\partial}{\partial Q}\pi^{\delta_Q}\left(Q^{\star}\left(R\right);s\right)d\sigma\left(s\right)>\int_{\mathrm{Supp}\sigma}\frac{\partial}{\partial Q}\pi^{\delta_Q}\left(Q^{M};s\right)d\sigma\left(s\right)=0.$$

This is a contradiction, hence $Q^M\leq Q^{\star\mathrm{PAB}}$. A similar argument applies to the case of $R^{\star\mathrm{PAB}}\leq R^M$. □

Remark 2. Since the optimal pay-as-bid auction sells a greater quantity at a lower reserve price, it is more efficient (generates greater total surplus) than either standard monopoly problem.

3.4 Comparative Statics: The Effect of Bidder Signal Distribution on Revenue

As an illustration of the separability theorem (Theorem 1), consider the issue whether the seller benefits from a mean-preserving spread of the distribution of bidders' signal.[6]

Proposition 5. **[Comparative Statics]** *If marginal values are increasing in the bidders' common signal, and are linear and additively separable in quantity and signal, then the seller's revenue in optimally designed pay-as-bid auction is increased by any mean-preserving spread of the distribution of bidders' signals.*

Proof. The assumptions of the proposition allow us to linearly renormalize the signal s so as to make it one-dimensional and represent the marginal revenue as $v(q;s) = s - \rho q$. We conduct the proof under the assumption that the distribution σ has no atoms; because any distribution σ can be approximated via atomless distributions and the seller maximization is continuous with respect to such approximations, imposing this assumption is without loss of generality. Let $\pi(s)$ be the equilibrium revenue associated with signal s and notice that Theorem 4 implies that π is differentiable in s except possibly at the threshold signal $s = \hat{s}$:

For $s < \hat{s}$, we have

$$\frac{d\pi}{ds} = \frac{d}{ds}\left[R^\star v^{-1}\left(R^\star; s\right)\right] = R^\star \frac{d}{ds}\left[\frac{1}{\rho}\left(s - R^\star\right)\right] = \frac{R^\star}{\rho}.$$

For $s > \hat{s}$, we have

$$\frac{d\pi}{ds} = \frac{d}{ds}\left[Q^\star v\left(Q^\star; s\right)\right] = Q^\star \frac{d}{ds}\left[s - \rho Q^\star\right] = Q^\star.$$

Thus, $\frac{d\pi}{ds}$ is piecewise constant and its value for $s < \hat{s}$ is strictly below its value for $s > \hat{s}$ because $\frac{R^\star}{\rho} < Q^\star$; the latter inequality being satisfied because Theorem 4 gives

$$R^\star = \frac{1}{2}\mathbb{E}\left[s|s < \hat{s}\right],$$
$$Q^\star = \frac{1}{2\rho}\mathbb{E}\left[s|s > \hat{s}\right]$$

(where the strict inequality follows from σ being atomless). Furthermore, at $s = \hat{s}$, $\frac{d\pi}{ds}$ has side derivatives as above. Defining $\frac{d\pi}{ds}|_{s=\hat{s}}$ to be a value weakly between the side derivatives, we find that $\frac{d\pi}{ds}$ that is convex.

[6] We explore this issue further in the example in Section 3.5.

Now, consider an alternate signal distribution σ'. As with σ we can assume that this distribution is atomless.[7] The above analysis implies that

$$\mathbb{E}[\pi] = \int_{\underline{s}}^{\overline{s}} \pi(s)\, d\sigma(s) = \pi(\underline{s}) + \int_{\underline{s}}^{\overline{s}} \frac{d\pi}{ds}(1-\sigma(s))\, ds,$$

where $\underline{s}, \overline{s}$ are such that $\operatorname{Supp}\sigma, \operatorname{Supp}\sigma' \subseteq [\underline{s}, \overline{s}]$; note that the value we set for $\frac{d\pi}{ds}|_{s=\hat{s}}$ doesn't matter because of the assumption that σ is atomless. The optimal revenue under distribution σ' is bounded below by the revenue obtained with the reserve $R^\star$ and quantity $Q^\star$ that are optimal for distribution σ, and thus the difference in optimal revenues is at least

$$\begin{aligned}
\mathbb{E}_{s\sim\sigma'}[\pi(s)] - \mathbb{E}_{s\sim\sigma}[\pi(s)] &= \int_{\underline{s}}^{\overline{s}} \mu\pi(s)(\sigma(s) - \sigma'(s))\, ds \\
&= \int_{\underline{s}}^{\overline{s}} \frac{R^\star}{\rho}(\sigma(s) - \sigma'(s))\, ds + \int_{\hat{s}}^{\overline{s}} \left(Q^\star - \frac{R^\star}{\rho}\right)(\sigma(s) - \sigma'(s))\, ds \\
&= \frac{R^\star}{\rho}\left(\mathbb{E}_{s\sim\sigma}[s] - \mathbb{E}_{s\sim\sigma'}[s]\right) + \left(Q^\star - \frac{R^\star}{\rho}\right)\left(\int_{\hat{s}}^{\overline{s}} \sigma(s) - \sigma'(s)\, ds\right).
\end{aligned}$$

If σ' is a mean-preserving spread of σ, the left-hand term is (definitionally) zero. Because, as noted above, $Q^\star > R^\star/\rho$, a mean-preserving spread σ' improves revenue if (but not necessarily only if)

$$\int_{\hat{s}}^{\overline{s}} \sigma(s) - \sigma'(s)\, ds > 0.$$

The latter condition is always satisfied when σ' is a mean-preserving spread of σ. □

Remark 3. Proposition 5 remains valid when the maximum feasible supply is below optimal monopoly supply, $Q^{\max} < Q^\star$. In this case, the optimal feasible quantity is $Q^{\max}$, and the optimal reserve is $R^{\max} < R^\star$. At the threshold signal $\hat{s}^{\max}$, it is the case that $R^{\max} = \hat{s}^{\max} - \rho Q^{\max}$, and $Q^{\max} > R^{\max}/\rho$. All derivations in Proposition 5 remain valid.

3.5 Example

Example 1. Take some constants $\rho, \underline{s}, \overline{s} > 0$, such that $\overline{s} > \underline{s} \geq \rho\overline{Q}/n$ and suppose that s is distributed uniformly on $(\underline{s}, \overline{s})$ and $v(q;s) = s - \rho q$ for some constant $\rho > 0$. Thus, $\varphi(R;s) = (s-R)/\rho$. For every relevant deterministic supply Q and reserve price R is then

[7]As later we will assume that σ' is a mean-preserving spread of σ, it is also important that we can approximate such σ and σ' by sequences of atomless distributions σ_k and σ'_k that converge to σ and σ', respectively, and that are such that σ'_k is a mean-preserving spread of σ_k. To find such sequences, we can, for instance, set σ_k to be $\sigma + U_{\frac{1}{k}}$ and σ'_k to be $\sigma' + U_{\frac{1}{k}}$ where $U_{\frac{1}{k}}$ is the uniform distribution on $\left[-\frac{1}{k}, \frac{1}{k}\right]$.

the unique cut-off $\tau = \tau(Q, R) = R + \rho Q/n$ such that[8]

$$R = v\left(\frac{Q}{n}; \tau\right) = \tau - \rho\frac{Q}{n}.$$

For all $s < \tau(Q, R)$ the seller sells quantity $\varphi(R; s) = n(s - R)/\rho$ at price R; for all $s > \tau(Q, R)$ the seller sells quantity Q at price $v(Q/n; s) = s - \rho Q/n$. Following Corollary 4, the seller's two-part maximization problem is[9]

$$\max_R \mathbb{E}_s\left[n\left(\frac{s-R}{\rho}\right)R\middle| s < \tau(Q^\star, R^\star)\right], \text{ and } \max_Q \mathbb{E}_s\left[\left(s - \frac{\rho Q}{n}\right)Q\middle| s > \tau(Q^\star, R^\star)\right].$$

This gives two equations,

$$\mathbb{E}_s\left[s|s < \tau(Q^\star, R^\star)\right] - 2R^\star = 0, \text{ and } n\mathbb{E}_s\left[s|s > \tau(Q^\star, R^\star)\right] - 2\rho Q^\star = 0.$$

Note that, after derivatives have been taken, we replace Q with $Q^\star$. Since $\mathbb{E}_s[s|s < \tau(Q^\star, R^\star)] = \underline{s} + (\tau(Q^\star, R^\star) - \underline{s})/2$ and $\mathbb{E}_s[s|s > \tau(Q^\star, R^\star)] = \overline{s} + (\tau(Q^\star, R^\star) - \overline{s})/2$, substituting in yields

$$2R^\star = \underline{s} + \frac{1}{2}\left(R^\star + \frac{1}{n}\rho Q^\star - \underline{s}\right), \text{ and } 2\frac{1}{n}\rho Q^\star = \overline{s} + \frac{1}{2}\left(R^\star + \frac{1}{n}\rho Q^\star - \overline{s}\right).$$

Then

$$R^\star = \frac{\overline{s} + 3\underline{s}}{8}, \text{ and } \frac{1}{n}Q^\star = \frac{3\overline{s} + \underline{s}}{8\rho}.$$

This immediately implies that $\tau = (\overline{s} + \underline{s})/2$.

Comparison to standard monopolists

The standard monopoly problems are straightforward. The quantity-monopoly problem is

$$\max_Q \mathbb{E}_s\left[Qv\left(\frac{Q}{n}; s\right)\right] = \max_Q Qv\left(\frac{Q}{n}; \mathbb{E}_s[s]\right) = \max_Q \left(\frac{\overline{s} + \underline{s}}{2} - \rho Q\right)Q.$$

Then optimal quantity is $Q^M = (\overline{s} + \underline{s})/(4\rho)$. The price-monopoly problem is

$$\max_R \mathbb{E}_s\left[nR\varphi(R; s)\right] \propto \max_R R\varphi(R; \mathbb{E}_s[s]) \propto \max_Q \left(\frac{\overline{s} + \underline{s}}{2} - R\right)R.$$

[8] Note that since signals are uni-dimensional and values are strictly monotone in signal, the sets $\underline{\mathcal{S}}(Q, R)$ and $\overline{\mathcal{S}}(Q, R)$ are uniquely identified with such a cut-off τ.

[9] Since the uniform distribution is massless, we can ignore the event $s = \tau$. Also, for expositional purposes we constrain attention to cases in which the seller's problem has an interior solution.

Then optimal price is $R^M = (\overline{s} + \underline{s})/4$.

$$\max_p n\mathbb{E}_s \left[\frac{1}{\rho} (s - p) p \right].$$

In the latter problem, the monopolist solves

$$\max_q \mathbb{E}_s \left[\left(s - \frac{\rho q}{n} \right) q \right].$$

Then $R^M = (\overline{s} + \underline{s})/4 > (\overline{s} + 3\underline{s})/8$, and $R^M > R^{\star\mathrm{PAB}}$; further, $Q^M = n(\overline{s} + \underline{s})/4\rho < n(3\overline{s}+\underline{s})/8\rho$, and $Q^M < Q^{\star\mathrm{PAB}}$. As shown in Proposition 4, the optimally designed pay-as-bid auction allocates a higher quantity at a lower (reserve) price than the classical monopolist's problem.

Effect of variance of bidder signal

The cutoff type is $\hat{s} = \frac{\overline{s}+\underline{s}}{2}$ and the expected revenue is $\frac{n}{2\rho}\left(\left(\frac{3\overline{s}+\underline{s}}{8}\right)^2 + \left(\frac{\overline{s}+3\underline{s}}{8}\right)^2\right)$, which we can express in terms of the mean $m = \frac{\overline{s}+\underline{s}}{2}$ and the variance $V = \frac{(\overline{s}-\underline{s})^2}{12}$ of the signal distribution:

$$\text{Expected Revenue} = \frac{n}{2\rho}\left(\frac{m^2}{2} + \frac{3V}{8}\right).$$

Expected revenue is directly proportional to the number of bidders, a somewhat surprising consequence of the linearity of the problem. The expected revenue is also increasing in the mean and variance of the signal distribution (Proposition 5) and decreasing in the steepness ρ of the marginal value function. The monotonicity in the variance of the distribution means that a mean-preserving spread induces gains on high types that outweigh the losses on low types even when the seller doesn't know the types; the is further able to limit the downside by setting the reserve.

3.6 Comparison to Monopoly

The separability of Theorem 4 allows us to compare optimally designed pay-as-bid reserve and supply to choices of a seller who sets the price (without optimizing over supply) and to the seller sets the supply allowing the price to be determined by Cournot-like market forces. In the context of the above example, the optimal price is half of the mean valuation for the initial unit, $p^{\mathrm{MONOP}} = \frac{m}{2}$, and the optimal supply is $q^{\mathrm{MONOP}} = \frac{m}{2\rho}n$ (that is the mean type utility on the optimal per-bidder supply is half of the utility on the initial unit), and hence

$$p^{\mathrm{MONOP}} > R^\star \quad \text{and} \quad q^{\mathrm{MONOP}} < Q^\star.$$

This is observed in Section 3.5. The optimally designed pay-as-bid auction allocates a higher quantity at a lower (reserve) price than the classical monopolist's problem. This feature arises from the ability of the pay-as-bid seller to hedge the two design parameters against one another. In Supplementary Appendix 3.3, we establish these comparisons more generally, thus showing how Theorem 4 contributes to the literature on whether price or quantity is a better instrument for achieving desired market outcomes.[10]

The separability of the pay-as-bid designer's problem shown in Theorem 4 sharply contrasts with problem faced by a designer of a uniform-price auction in PW Section 6. In the uniform-price auction equilibrium bids are not (in general) unique, the strict monotonicity of bid in value cannot be assured, and it is not necessarily the case that uniform-price auctions can be optimized in a separable manner.

4 Large Market Revenue Equivalence

Proposition 6. **[Large Market Revenue Equivalence]** *If per-capita aggregate supply F^{μ} has full support, equilibrium expected revenue in the pay-as-bid and uniform-price auctions converges to the expected revenue in a uniform-price auction with truthful reporting as the number of bidders grows large.*

Proof. Equilibrium bids in the pay-as-bid auction, given in PW Theorem 3, converge as $n \nearrow \infty$; the same is true of equilibrium bids in the uniform-price auction, given in PW Lemma 15. Moreover, equilibrium bids in the uniform-price auction with full support converge to truthful reporting.

It remains to be seen that equilibrium per capita expected revenue in the pay-as-bid auction with many bidders is equal to per capita expected revenue in the uniform-price auction with truthful reporting. Letting $q(Q; R, s) = \min\{Q, v^{-1}(R; s)\}$, we establish the claim pointwise for each signal s,

$$\int_0^{\overline{Q}^{\mu}} q\left(Q; R, s\right) v\left(q\left(Q; R, s\right); s\right) dF^{\text{per capita}}\left(Q\right)$$
$$= \int_0^{\overline{Q}^{\mu}} \int_0^{Q} \int_{Q'}^{\overline{Q}^{\mu}} v\left(q\left(x; R, s\right); s\right) dF^{Q', \text{per capita}}\left(x\right) dQ' dF^{\text{per capita}}\left(Q\right).$$

[10]Weitzman [1974] obtains conditions under which price or quantity regulation is preferred under stochastic demand and supply; Roberts and Spence [1976] find that a three-part system involving permits, penalties, and repurchase is preferable to any single-instrument system.

Integration by parts of the innermost integral rearranges the right-hand side to

$$\int_0^{\overline{Q}^\mu} \int_0^Q v\left(q\left(Q;R,s\right);s\right) + \int_{Q'}^{\overline{Q}^\mu} 1\left[x < v^{-1}\left(R;s\right)\right] v_q\left(q\left(x;R,s\right);s\right) \frac{1 - F^{\text{per capita}}\left(x\right)}{1 - F^{\text{per capita}}\left(Q'\right)} dQ' dF^{\text{per capita}}\left(Q\right).$$

Subsequent integration by parts of the second integral rearranges the right-hand side to

$$\int_0^{\overline{Q}^\mu} Qv\left(q\left(Q;R,s\right);s\right) + Q\int_Q^{\overline{Q}^\mu} 1\left[x < v^{-1}\left(R;s\right)\right] v_q\left(q\left(x;R,s\right);s\right) \frac{1 - F^{\text{per capita}}\left(x\right)}{1 - F^{\text{per capita}}\left(Q\right)} dx$$
$$- \int_0^Q Q' \int_{Q'}^{\overline{Q}^\mu} 1\left[x < v^{-1}\left(R;s\right)\right] v_q\left(q\left(x;R,s\right);s\right) \frac{\left(1 - F^{\text{per capita}}\left(x\right)\right) f^{\text{per capita}}\left(Q'\right)}{\left(1 - F^{\text{per capita}}\left(Q'\right)\right)^2} dx dQ' dF^{\text{per capita}}\left(Q\right).$$

Because $\int_0^{\overline{Q}^\mu} Qv\left(q\left(Q;R,s\right);s\right) dF^\mu\left(Q\right)$ is expected revenue in the uniform-price auction with truthful reporting, it is sufficient to show that

$$\int_0^{\overline{Q}^\mu} Q \int_Q^{\overline{Q}^\mu} 1\left[x < v^{-1}\left(R;s\right)\right] v_q\left(q\left(x;R,s\right);s\right) \frac{1 - F^{\text{per capita}}\left(x\right)}{1 - F^{\text{per capita}}\left(Q\right)} dx dF^{\text{per capita}}\left(Q\right)$$
$$= \int_0^{\overline{Q}^\mu} \int_0^Q Q' \int_{Q'}^{\overline{Q}^\mu} 1\left[x < v^{-1}\left(R;s\right)\right] v_q\left(q\left(x;R,s\right);s\right) \frac{\left(1 - F^{\text{per capita}}\left(x\right)\right) f^{\text{per capita}}\left(Q'\right)}{\left(1 - F^{\text{per capita}}\left(Q'\right)\right)^2} dx dQ' dF^{\text{per capita}}\left(Q\right).$$

To streamline exposition, define

$$J\left(Q\right) = \int_Q^{\overline{Q}^\mu} 1\left[x < v^{-1}\left(R;s\right)\right] v_q\left(q\left(x;R,s\right);s\right) \frac{1 - F^{\text{per capita}}\left(x\right)}{1 - F^{\text{per capita}}\left(Q\right)} dx.$$

Then the desired equality is

$$\int_0^{\overline{Q}^\mu} QJ\left(Q\right) dF^\mu\left(Q\right) = \int_0^{\overline{Q}^\mu} \int_0^Q Q' J\left(Q'\right) \frac{f^{\text{per capita}}\left(Q'\right)}{1 - F^{\text{per capita}}\left(Q'\right)} dQ' dF^{\text{per capita}}\left(Q\right).$$

Integration by parts of the outer integral rearranges the right-hand side to

$$\int_0^{\overline{Q}^\mu} QJ\left(Q\right) f^{\text{per capita}}\left(Q\right) dQ = \int_0^{\overline{Q}^\mu} QJ\left(Q\right) dF^{\text{per capita}}\left(Q\right).$$

Then per capita expected revenue in a large pay-as-bid auction is identical to per capita expected revenue in a large uniform-price auction with truthful reporting. $\square$

5 Increasing Marginal Costs

The proof of PW Theorem 5 remains valid for the profit maximization problem of a seller facing increasing marginal costs. Let $C(Q)$ be the seller's cost of supplying quantity Q, and assume that $c(Q) = dC(Q)/dQ$ is positive and weakly increasing. Equation (10) for expected profits in the proof of PW Theorem 5 must be adjusted to

$$\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s \int_0^{\overline{Q}} \int_0^{Q^R(y,s)} p\left(x; R, s\right) - c\left(x\right) dx dF\left(y\right).$$

Subsequent integration by parts remains valid, and PW equation (10) becomes

$$\mathbb{E}\left[\pi^F\right] = \mathbb{E}_s \int_0^{\overline{Q}^R(s)} \left(1 - F\left(y\right)\right) \left(v\left(\frac{1}{n}y; s\right) - c\left(y\right)\right) + \left(1 - F\left(y\right)\right)^{\frac{1}{n}} \int_y^{\overline{Q}^R(s)} \frac{1}{n} v_q \left(\frac{1}{n}x; s\right) \left(1 - F\left(x\right)\right)^{\frac{n-1}{n}} dxd$$

As before, letting $\pi^{\delta_q}(q; s, c)$ be monopoly profits when quantity q is sold to type s given marginal cost curve c, we obtain

$$\mathbb{E}\left[\pi^F\right] \leq \mathbb{E}_s \left[\int_0^{\overline{Q}} \pi^{\delta_{Q^R(x;s)}} \left(Q^R\left(x, s\right); s, c\right) dF\left(x\right)\right].$$

The remainder of the proof is immediate.

6 Asymmetric Information

Now suppose that bidder i's private information is $s_i = (s, \theta_i)$, where s is a common signal known to all bidders and θ_i is idiosyncratic and privately known only to bidder i; notice that we do not require that θ_i and θ_j are independent nor do we require that they are identically distributed.[11] For the sake of expositional simplicity we normalize the signals so that each idiosyncratic signal θ_i has identical support containing 0, and we treat the case of all idiosyncratic signals taking value 0 as the benchmark common signal case.[12] Letting $\mathcal{S}_s = \operatorname{Supp} s$ and $\mathcal{S}_\theta = \operatorname{Supp} \theta_i$, we assume that bidder information has full support, so that $\operatorname{Supp} \theta_i|_{s,\theta_{-i}} = \mathcal{S}_\theta$, but otherwise there are no distributional assumptions on s, θ_i, or their interrelation. The proof of PW Theorem 1 can be extended to the asymmetric information

[11] The separation of signals into common and idiosyncratic components is convenient but inessential; signals can always be separated in this way and the separation simplifies the definition of bounded informational asymmetry as well as comparisons to the benchmark model with only a common signal.

[12] Our results (and the definition of bounded asymmetry) allow heterogenous $v^i(\cdot; \cdot, \cdot)$, provided $v^i(\cdot; \cdot, 0)$ does not depend on i.

environment.[13] We then obtain

Lemma 1. [A Bound on Market Price] *In any mixed-strategy equilibrium of the pay-as-bid auction, for any signal profile $(s, (\theta_1, \ldots, \theta_n))$ all realizations of the market clearing price for the effective maximum quantity $\overline{Q}^R$ are bounded between the smallest and largest marginal value at the per-capita effective maximum quantity,*

$$\min_i \operatorname{ess\,inf}_{\tilde{\theta}_i} v^i \left(\frac{1}{n} \overline{Q}^R; s, \tilde{\theta}_i \right) \leq p \left(\overline{Q}^R; s, \theta \right) \leq \max_i \operatorname{ess\,sup}_{\tilde{\theta}_i} v^i \left(\frac{1}{n} \overline{Q}^R; s, \tilde{\theta}_i \right).$$

We now discuss the robustness of our results to the presence of informational asymmetries. To do this, we first define a notion of boundness of informational asymmetry.

Definition 2. For $\delta \geq 0$, we say that informational asymmetry is *δ-bounded* if, for all $(s, \theta) \in \mathcal{S}_s \times \mathcal{S}_\theta$, $\sup_q |v(q; s, 0) - v(q; s, \theta)| \leq \delta$.

When marginal values v are bounded above by $\delta \geq 0$, informational asymmetries are δ-bounded. Thus δ-bounded informational asymmetry is a relatively weak restriction if δ is large; at the same time our results become tight only as δ becomes small.

For bounded asymmetries, we show that the expected revenue in any equilibrium of an optimal pay-as-bid auction—that is, pay-as-bid with optimal supply and reserve price—with asymmetric private information is *nearly above* the expected revenue in the unique equilibrium of the optimized auction when bidders' information is symmetric: expected revenue is above the revenue in the uniform price auction, decreased by $\delta Q^\star$, where $Q^\star$ is the optimal supply in the symmetric information environment (with $\theta_i = 0$). We analogously define *nearly below* and *nearly indifferent.*

Theorem 2. [A Bound on Revenue Loss from Informational Asymmetry] *Suppose that asymmetry is δ-bounded. Then, the expected revenue in any equilibrium of the optimal pay-as-bid auction is nearly above the expected revenue in the unique equilibrium of the optimal pay-as-bid auction with symmetric bidder information (in which $\theta_i = 0$).*

This theorem implies that small informational asymmetries do not dramatically reduce the seller's revenue below the symmetric-information benchmark. This implication of our theorem is not a simple limit result. First, in environments for which purification results have been proven, a limit of equilibria as we decrease the import of idiosyncratic signals is a mixed-strategy equilibrium in the limit environment, but in Theorem 2 we bound the revenue from below by a pure-strategy equilibrium in the limit environment. Second, there

[13]For more details see Pycia and Woodward [2020] and Pycia and Woodward [2023a].

are so far no purification results for such infinitely-dimensional discontinuous games as pay-as-bid auctions. We are able to establish the above theorem because of our earlier results showing that when bidders' information is symmetric then optimal supply is deterministic. The following result plays a key role in its proof.

Lemma 2. *In a pay-as-bid auction with deterministic supply and reserve R, the expected revenue in any equilibrium with δ-bounded asymmetric private information is nearly above the expected revenue of pay-as-bid with same supply and reserve* $\max\{R-\delta, 0\}$ *in the unique equilibrium when bidders' information is symmetric.*

The lemma follows from Lemma 1, in which we establish that the market clearing price is bounded below by the lowest marginal value $v(\cdot;\cdot,\cdot)$ of per capita supply. When $v(\cdot;\cdot,\cdot)$ is within δ of $v(\cdot;\cdot,0)$, this lowest marginal value of per capita supply is weakly above $v\left(\frac{Q}{n};\cdot,0\right)-\delta$. By setting the deterministic quantity Q at optimal value at symmetric information and lowering the reserve price by δ with respect to optimal reserve R at symmetric information, the seller can sell at least the same quantity of the good. When bidders are asymmetrically informed, the per-unit price is bounded below by $\max\{v\left(\frac{Q}{n};s,0\right)-\delta, R-\delta\}$, while $\max\{v\left(\frac{Q}{n};s,0\right), R\}$ is the per-unit revenue in the unique equilibrium when bidders' information is symmetric and equals $(s,0)$. Thus Lemma 2 obtains.

Theorem 2 then follows from Lemma 2 because our transparency results guarantee that deterministic supply is optimal in the symmetric information benchmark, and because allowing the seller to re-optimize supply in the presence of informational asymmetries weakly improves revenue.

6.1 Pay-as-Bid vs. Uniform-Price

We now show that the revenue comparison results of the main text continue to hold for asymmetrically-informed bidders when the informational asymmetry is small. The comparison requires understanding the behavior of asymetrically-informed bidders in pay-as-bid auctions and uniform-price auctions as well as the design response of the seller. The logic developed above, giving an approximate revenue bound in the pay-as-bid auction when the informational asymmetry is δ-bounded, also applies to the uniform-price auction, with the exception that equilibrium may be nonunique. In the uniform-price auction with private information, every equilibrium generates revenue that is weakly below quantity times the maximum marginal value for per capita supply.

Lemma 3. *Fix a reserve price and supply distribution. In a uniform-price auction, the expected revenue in an equilibrium with asymmetrically-informed bidders is nearly below the expected revenue under truthful bidding with symmetrically-informed bidders.*

Note that revenue under truthful bidding is an upper bound on equilibrium revenue.

Proof. In the uniform-price auction, the parts of bids that determine the equilibrium market-clearing price are bounded above by truthful reporting. Let Q be a realization of supply; notice that Q is bounded above by the maximum supply $\bar{Q}$. Conditional on this supply realization, for any fixed $\delta > 0$, the expected revenue under asymmetric information that is within δ of $v(\cdot; s, 0)$ is bounded above by $Q\left[v\left(\frac{Q}{n}; s, 0\right) + \delta\right] \leq Qv\left(\frac{Q}{n}; s, 0\right) + \delta\bar{Q}$. The result follows because $Qv\left(\frac{Q}{n}; s, 0\right)$ is the revenue under truthful bidding, conditional on the common signal being s when bidders are symmetrically informed. □

The above two lemmas establish an approximate version of pay-as-bid revenue dominance. With slight abuse of terminology, in the following theorem we say that the pay-as-bid expected revenue is *nearly above* uniform-price expected revenue when the difference between the two is bounded from below by $-2\delta Q^\star$ (rather than $-\delta Q^\star$ as before), where $Q^\star$ is the optimal supply in the symmetric information environment.

Theorem 3. [Approximate Revenue Dominance of Optimal Pay-as-Bid] *With optimal reserve price and supply, the expected revenue in the pay-as-bid auction is nearly above expected revenue in the uniform-price auction.*

In particular, for any $\varepsilon > 0$, if the informational asymmetry is $\frac{\varepsilon}{2Q^*}$-small, where Q^* is the optimal supply in the symmetric information environment, then $\mathbb{E}\left[\pi^{\mathrm{PAB}}\right] \geq \mathbb{E}\left[\pi^{\mathrm{UP}}\right] - \varepsilon$. This theorem further implies that the analogue of Corollary 5 from Pycia and Woodward [2023b] holds true:

Corollary 5. *In any perfect Bayesian equilibrium of the auction design game, the seller either implements a pay-as-bid auction or is the absolute value of the difference of the expected revenues of the two formats is bounded by $2\delta Q^\star$.*

We show below that deterministic supply is nearly optimal, but the optimal supply does not need to be deterministic. Still, an analogue of the above theorem obtains for potentially suboptimally-designed auctions as long as they are deterministic.

Theorem 4. [Approximate Revenue Dominance of Pay-as-Bid with Deterministic Supply] *Given any deterministic supply Q and reserve price R, expected revenue in the pay-as-bid auction is nearly above expected revenue in the uniform-price auction. Moreover, the seller is nearly indifferent between any equilibrium of the pay-as-bid auction and any revenue-maximizing equilibrium of the uniform-price auction.*

Proof. The analogue of Lemma 2 obtains for (not necessarily optimal) supply and reserve as long as they are deterministic; the proof follows the same steps. The first statement follows then from this deterministic analogue of Lemma 2 and from Lemma 3. To prove the second statement, consider a uniform-price auction where bids, conditional on common signal s, are bounded below by $\underline{b}(s) = \max\{R, \operatorname{ess\,inf}_{\zeta|s} v(Q/n;\zeta)\}$: bidding below $\underline{b}(s)$ cannot yield additional quantity, and by construction, when $b \geq \underline{b}(s)$ the marginal value for all units obtained is weakly positive. It follows that there is an equilibrium in which bids are at least $\underline{b}(s)$, and the second claim follows. $\square$

Remark 4. The analogue of this theorem continues to hold if there is small uncertainty over supply. Without the asymmetry of information, this point follows from the continuity of optimal bidding strategies in pay-as-bid with respect to supply because our bound on uniform-price revenue is in terms of truthful bidding. The asymmetry of information does not affect the uniform-price bound, and we can control the change in the lower bound on pay-as-bid revenue via the price bound of Theorem 1.

The above two theorems tell us that, while the uniform-price auction might generate greater revenue than a pay-as-bid auction, this difference will not be large without a significant informational asymmetry among bidders or significant randomness in supply. Thus, a version of PW Corollary 5 holds in the presence of informational asymmetries: the seller either strictly prefers a pay-as-bid auction or is approximately indifferent between the pay-as-bid and uniform-price auctions.

6.2 Approximate Optimality of Transparency

The analysis of elastic supply and mixed-strategy equilibria in PW Appendix A shows that if buyers' values are regular, a deterministic supply curve maximizes the seller's revenue. In this subsection we apply this analysis to the design of optimal pay-as-bid auctions in the presence of small informational asymmetries.

Definition 3. [Regular Demand] Let $\mathcal{S} = \{(p^\star, q^\star) \colon \exists s,\ p^\star \in \arg\max_p pv^{-1}(p;s),\ q^\star = v^{-1}(p;s)\}$ be the set of optimal monopoly price-quantity pairs. Bidder values are *regular* if, for any $(p,q), (p',q') \in \mathcal{S}$, the inequality $p' < p$ implies $q' < q$.

Values are regular if the monopolist's optimal price and quantity are in monotone correspondence.[14] When values are increasing in signal s and v^{-1} is differentiable, demand is

[14]Recall that we do not make any assumptions on the bidders' type space, and in particular we do not require that demand increases with type.

regular when $p + v^{-1}(p;s)/v_p^{-1}(p;s)$ is increasing in s.[15] Thus our regularity condition is similar to the regularity condition in [Myerson, 1981]. When values are regular, the auctioneer can use an elastic supply curve to screen for bidder signal s, and a deterministic elastic supply curve maximizes the seller's revenue.

Theorem 5. [Approximate Optimality of Transparency] *Suppose buyers' values are regular. For any $\varepsilon > 0$, there is $\delta > 0$ such that if informational asymmetry is δ-small then there is a deterministic elastic supply curve S that is approximately optimal: $\mathbb{E}\left[\pi^S\right] \geq \mathbb{E}\left[\pi^K\right] - \varepsilon$ for any, potentially stochastic, elastic supply K.*

PW Appendix H (Lemma 17) implies that when the bidder's private information $s_i = (s, \theta_i)$ is known to the seller, the seller's revenue is strictly higher when a deterministic quantity is sold than when the buyer faces any randomness in residual supply. Then revenue with asymmetric information is bounded above by the revenue the seller would obtain with optimal monopoly supply targeted to each bidder's private information s_i.

Monopoly revenue is strictly increasing in marginal value. Then when asymmetric information is δ-small,

$$\max_q q \cdot \left[v\left(q;s,0\right) - \delta\right] \leq \max_q q \cdot v\left(q;s,\theta_i\right) \leq \max_q q \cdot \left[v\left(q;s,0\right) + \delta\right].$$

Furthermore, if the seller knows s but not θ_i, we may bound optimal expected monopoly profits below by

$$\max_q q \cdot \left[v\left(q;s,0\right) - \delta\right] \leq \max_q \mathbb{E}\left[q \cdot v\left(q;s,\theta_i\right)\right].$$

When demand is regular, it follows that expected revenue under deterministic elastic supply cannot be significantly worse than expected revenue under optimal elastic supply, where the difference is no greater than $2\delta Q^{\max}$.

7 Relationship to dynamic oligopoly

PW's analysis of pay-as-bid auctions can be reinterpreted as a model of dynamic oligopolistic competition among sellers who at each moment of time compete à la Bertrand for sales and who are uncertain how many more buyers are yet to arrive. Prior sales determine the production costs for subsequent sales, thus the sellers need to balance current profits with

[15] To maximize profits, $d[pv^{-1}(p;s)]/dp = 0$, implying $p + v^{-1}(p;s)/v_p^{-1}(p;s) = 0$. If the left-hand side is increasing in s, then $p^\star$ is increasing in s. To have quantity also increasing in s, we need $d[qv(q;s)]dq = 0$, or $qv_q(q;s) + v(q;s) = 0$. Under monopoly, $q = v^{-1}(p;s)$ and $p = v(q;s)$, and the conditions for monotonicity in price and in quantity are equivalent.

the change in production costs in the future. This methodological link between pay-as-bid auctions and dynamic oligopolistic competition is new, and we develop it in follow up work.[16]

References

Jeremy Bulow and Paul Klemperer. Auctions versus negotiations. *The American Economic Review*, 86(1):180–194, 1996. ISSN 00028282.

Raymond Deneckere and James Peck. Dynamic competition with random demand and costless search: A theory of price posting. *Econometrica*, 80(3):1185–1247, May 2012.

Paul D Klemperer and Margaret A Meyer. Supply function equilibria in oligopoly under uncertainty. *Econometrica*, pages 1243–1277, 1989.

Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.

Alessandro Pavan, Ilya Segal, and Juuso Toikka. Dynamic mechanism design: A myersonian approach. *Econometrica*, 82(2):601–653, 2014.

Marek Pycia and Kyle Woodward. Auctions of homogeneous goods: A case for pay-as-bid. 2020.

Marek Pycia and Kyle Woodward. Pollution permits: Efficiency by design. 2023a.

Marek Pycia and Kyle Woodward. A case for pay-as-bid auctions. 2023b.

Marc J Roberts and Michael Spence. Effluent charges and licenses under uncertainty. *Journal of Public Economics*, 5(3):193–208, 1976.

Martin L Weitzman. Prices vs. quantities. *The Review of Economic Studies*, 41(4):477–491, 1974.

[16]The oligopolistic sellers uncertain of future demands correspond to bidders in the pay-as-bid auction, and sellers' costs correspond to bidders' values. For prior studies of dynamic competition see e.g. Deneckere and Peck [2012]; while they study competition among a continuum of sellers, the pay-as-bid-based approach allows for the strategic interaction between a finite number of sellers. The other canonical multi-unit auction format, the uniform-price auction, was earlier interpreted in terms of static oligopolistic competition by Klemperer and Meyer [1989]. For dynamic mechanisms more broadly, cf., e.g., Pavan et al. [2014].