Auctions of Homogeneous Goods: A Case for Pay-as-Bid

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Abstract

The pay-as-bid (or discriminatory) auction is a prominent format for selling homogeneous goods such as treasury securities and commodities. We prove the uniqueness of its pure-strategy Bayesian Nash equilibrium and establish a tractable representation of equilibrium bids. Building on these results we analyze the optimal design of pay-as-bid auctions, as well as uniform-price auctions (the main alternative auction format). We show that supply transparency and full disclosure are optimal in pay-as-bid, though not necessarily in uniform-price; pay-as-bid is revenue dominant and might be welfare dominant; and, under assumptions commonly imposed in empirical work, the two formats are revenue and welfare equivalent.


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1 Introduction

Each year, securities and commodities worth trillions of dollars are allocated through multi-unit auctions. Pay as bid is one of two main auction formats for these sales; the other format being uniform-price. Pay-as-bid is often used to sell treasury securities and to distribute electricity generation. It is also used in government operations such as recent large-scale asset purchases in the U.S. (quantitative easing), and it is implicitly run in financial markets when limit orders are followed by a market order.\footnote{Pay-as-bid auctions are also referred to as discriminatory, or multiple-price auctions. OECD [2021] finds that 25 of 37 countries surveyed allocate securities via pay-as-bid auctions; Brenner et al. [2009] find that 33 of 48 countries surveyed use pay-as-bid. Del Río [2017] finds that 27 of 31 markets surveyed distribute electricity generation via pay-as-bid auction (see also Maurer and Barroso [2011]). In all these studies most of the remaining markets are cleared by uniform-price auction and in some markets both formats are used. For financial markets, see, e.g., Glosten [1994].} Despite their economic importance, relatively little is known about equilibrium behavior in pay-as-bid auctions. Accordingly, little is known about the design problem faced by the pay-as-bid auctioneer: for instance, what is the optimal reserve price, and how does transparency about supply affect the seller’s revenue? Furthermore, empirical studies find rough revenue equivalence of pay-as-bid and uniform-price auctions, posing an intriguing puzzle for theoretical research.\footnote{Pay-as-bid auction equilibria have been constructed in parameterized environments; see our discussion below. The empirical literature on multi-unit auctions provides no definitive result on which auction format raises more revenue and Hortaçsu et al. [2018] posit that this is potentially because bidders retain little surplus.}

This paper addresses these open questions focusing on environments in which the bidders are symmetrically informed, an assumption that is approximately satisfied in many multi-unit auction environments.\footnote{We show in companion work [Pycia and Woodward, 2022a] that our results are robust to allowing for small informational asymmetries.} For example, treasury securities often have close substitutes whose prices are known and the forward contracts based on the current issue are traded ahead of the auction in the forward markets, providing bidders with substantial information about each others’ valuations. The U.K. Debt Management Office highlighted this feature of the informational environment in which it sells the British gilt-edged securities, by noting in its guide to UK securities that:

“There are often similar gilts already in the market to allow ease of pricing [...] This suggests that bidders are not significantly deterred from participation by not knowing what the rest of the market’s valuation of the gilts on offer is” [UK DMO, 2012].

In empirical analyses, Hortaçsu et al. [2018] argue that bidders in U.S. Treasury auctions of short-term securities are nearly symmetrically informed, Armantier and Lafhel [2009] argue...
that bidders in Bank of Canada auctions are essentially symmetric, and Hattori and Takahashi [2022] argue the same for bidders for Japanese government bonds. Our results allow any informational asymmetry between the seller and the bidders. The difference between seller’s and bidders’ information is typical of the problem we study because the seller designs the auction before—usually substantially before—the bidders submit their bids; the seller may also want to fix a single design for multiple auctions. We allow for uncertainty of the total supply available for auction as exogenous supply uncertainty is a feature of some securities auctions, e.g. in the United States [TreasuryDirect, 2022] and Japan [Hattori and Takahashi, 2022].

Our first contribution lies in expanding the bounds within which bidding behavior in the pay-as-bid auction is tractable: we allow an arbitrary number of bidders and general demands. We determine the lowest equilibrium market-clearing price. This price obtains and our subsequent design insights are valid whether or not we allow mixed strategy-equilibria, but our theory of equilibrium bidding in pay-as-bid auctions focuses on pure-strategy equilibria. We prove that pure-strategy equilibrium is unique—in contrast with the substantial equilibrium multiplicity present in the uniform-price auction [Wilson, 1979, Klemperer and Meyer, 1989, Wang and Zender, 2002]—and that bids have an unexpectedly tractable closed-form representation. We also establish a sufficient condition for the existence of equilibrium; our condition is satisfied when, e.g., there are sufficiently many bidders and their marginal values are smooth.

Our main design result establishes the revenue-optimality of transparency in setting supply in pay-as-bid auctions: revenue in the unique pure-strategy equilibrium is maximized when supply is deterministic. Thus determining the optimal supply distribution is equivalent to the simpler problem of a monopolist who sets a price and a quantity cap. Because in some treasury auctions the distribution of supply is partially determined by the demand from non-competitive bidders, and treasuries and central banks retain only partial ability to influence supply distributions, we also address the question of how much data on non-

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4Our result that in absence of substantive uncertainty, bids in the pay-as-bid auction are approximately flat, provides a test of the symmetric information assumption. The flatness of bids has been observed in treasury auctions in several countries, see Section 6 for a discussion. While our assumptions are borne out in some important multi-unit auctions, they are not satisfied in others: for example, Armantier and Sbaï [2009] argue that bidders in French debt auctions are asymmetrically informed.

5A corresponding uncertainty over auctioned demand is a feature of many spot electricity auctions, where demand is determined by the current state of electricity usage; cf. Federico and Rahman [2003], Hortaçsu and Puller [2008], and U.S. Federal Energy Regulatory Commission [2020], among others.

6While in this discussion we focus on the seller setting reserve price and distribution of supply in pay as bid, in Appendix A we show that our insights also extend to the case when the seller can set a distribution over elastic supply curves; this extension relies on a Myerson-like regularity assumption imposed on bidders’ values.
competitive bids should be released to competitive bidders. We show that the seller wants to commit to reveal the realization of supply before bids are submitted. Furthermore, it is optimal for the seller to inform bidders of the supply available, regardless of her ability to influence its distribution. The principle of transparent design we establish for pay-as-bid auctions simplifies their design in a way that does not carry over to the design of uniform-price auctions; for the latter we show that neither deterministic supply nor information release are necessarily revenue-optimal.\footnote{The reason for this failure is the multiplicity of equilibria in uniform-price auctions. Although there is a uniform-price auction with deterministic supply admitting a revenue-optimal equilibrium, these auctions also admit low- and zero-revenue equilibria; see, e.g., Kremer and Nyborg [2004], LiCalzi and Pavan [2005], McAdams [2007], Burkett and Woodward [2020b], and Marszalec et al. [2020]. Depending on the auctioneer’s concern about equilibrium selection, anticipated revenue may improve with some randomization, see our Lemma 1. Of course, revenue may be only one of auctioneer’s objectives.}

Relying on our results on equilibrium bidding and supply transparency in the pay-as-bid auction, we are able to compare revenues and welfare in optimally-designed pay-as-bid and uniform-price auctions. We prove that the pay-as-bid format always raises weakly higher revenue, while the welfare comparison depends on equilibrium selection in uniform-price auction, which in general allows for multiple equilibria. In particular, a revenue-maximizing seller would run a uniform-price auction only if its revenue equaled that of pay as bid. Major empirical studies comparing revenues between pay-as-bid and uniform-price auctions consider strategy profiles in which bidders in uniform-price bid truthfully for the marginal unit.\footnote{See e.g. Hortaçsu and McAdams [2010] and Marszalec [2017], and our discussion below. An exception is the constrained strategic equilibrium approach developed by Armantier et al. [2008] and applied by Armantier and Sbai [2006], among others.} Truthful bidding can be—but does not need to be—an equilibrium of an optimally-designed uniform-price auction; if bidders bid truthfully, then our results show that both revenue and welfare are the same across the pay-as-bid and uniform-price auction formats. Thus our results provide a theoretical explanation for the approximate revenue equivalence found in some empirical work.

Before situating our results in the rich related literature, we describe how the pay-as-bid auction operates. First, the bidders submit bids for each infinitesimal unit of the good. Then, the supply is realized, and the auctioneer (or, the seller) allocates the first infinitesimal unit to the bidder who submitted the highest bid, then the second infinitesimal unit to the bidder who submitted the second-highest bid, etc.\footnote{To fully-specify the auction we need to specify a tie-breaking rule; we adopt the standard tie-breaking rule, pro-rata on the margin, but our theory of equilibrium bidding does not hinge on this choice. This is in contrast to uniform-price auction, where tie-breaking matters; see Kremer and Nyborg [2004].} Each bidder pays her bid for each unit she obtains. The monotonic nature of how units are allocated implies that a collection of bids a bidder submitted can be equivalently described as a reported demand curve that is weakly-
decreasing in quantity, but not necessarily continuous; the ultimate allocation resembles that of a classical Walrasian market, in which supply equals demand at a market-clearing price. We study pure-strategy Bayesian Nash equilibria of this auction.\textsuperscript{10}

We establish a bound on the equilibrium market clearing price in terms of bidders’ marginal values. The special cases of our bound are implicit in the equilibrium constructions in the parametric examples of pay-as-bid we discuss below, but ours is the first bound on all pure-strategy equilibria, as well as the first bound that allows for mixed-strategy equilibria.\textsuperscript{11} The bound plays a crucial role in our analysis of equilibria and in our revenue comparisons.

We provide two sufficient conditions for equilibrium existence: a complex condition that is more general but difficult to analyze, and a simple condition that is less general but straightforward. Our simple condition reduces the existence question to checking optimization properties pointwise. It is satisfied, for instance, in the linear-Pareto settings analyzed by the prior literature discussed above, as well as for convex marginal values and for any distribution of supply provided there are sufficiently many bidders.\textsuperscript{12} There is a large literature on equilibrium existence in pay-as-bid auctions. In symmetric-information settings, in addition to the contributions discussed above, Holmberg [2009] proves the existence of equilibrium when the distribution of supply has a decreasing hazard rate, and recognizes the possibility that (pure-strategy) equilibrium may not exist.\textsuperscript{13} Our sufficient condition for existence encompasses the prior conditions and is substantially milder. In asymmetric information settings, Athey [2001], McAdams [2003], and Reny [2011] have shown that equilibrium exists in multi-unit (discrete) pay-as-bid auctions, and Woodward [2019] established existence in the asymmetric-information analogue of the divisible-good model we study.\textsuperscript{14} A key difference between the results in these papers and ours is that the presence of private information allows the purification of mixed-strategy equilibria; such purification is not possible in the symmetric-information setting. Our existence conditions are consequences of

\textsuperscript{10}In equilibrium, each bidder responds to the stochastic residual supply (that is, the supply given the bids of the remaining bidders). Effectively, the bidder is picking a point on each residual supply curve. In determining her best response, a bidder needs to keep in mind that: (i) the bid that is marginal if a particular residual supply curve is realized is paid not only when it is marginal, but also in any other state of nature that results in a larger allocation, and hence the bidder faces tradeoffs across these different states of nature; and (ii) bid curves need to be weakly monotonic in quantity, potentially a binding constraint.

\textsuperscript{11}A different bound, in terms of competitive markets, was obtained by Swinkels [1999] for large economies. Our bound applies is valid in all finite markets.

\textsuperscript{12}For many distributions of interest our condition is also satisfied with relatively few bidders; we provide examples in Section 3.

\textsuperscript{13}See Genc [2009] and Anderson et al. [2013] for discussions of potential problems with equilibrium existence.

\textsuperscript{14}For equilibrium existence in multi-unit auctions, see also Břeský [1999], Jackson et al. [2002], Reny and Zamir [2004], Jackson and Swinkels [2005], McAdams [2006], Břeský [2008], and Kastl [2012]. Milgrom and Weber [1985] show existence of mixed-strategy equilibria.
our uniqueness and representation theorems, and (unlike general existence results) are not independent of the form of equilibrium.\(^\text{15}\)

Our theorems establishing the existence and uniqueness of pure-strategy Bayesian Nash equilibrium in pay-as-bid auctions are reassuring for sellers using the pay-as-bid format; indeed, there are well-known problems posed by multiplicity of equilibria in other multi-unit auction formats.\(^\text{16}\) Uniqueness is also important for the empirical study of pay-as-bid auctions. Estimation strategies based on the first-order conditions, or the Euler equation, rely on agents playing comparable equilibria across auctions in the data (Février et al. [2002], Hortaçsu and McAdams [2010], Hortaçsu and Kastl [2012], and Cassola et al. [2013]).\(^\text{17}\) Equilibrium uniqueness plays an even larger role in the study of counterfactuals (see Armantier and Sbaï [2006]).\(^\text{18}\)

Uniqueness was studied by Wang and Zender [2002] who prove the uniqueness of “nice” equilibria under strong parametric assumptions on utilities and distributions. Assuming that marginal values are linear and that supply is drawn from an unbounded Pareto distribution, they analyzed symmetric equilibria in which bids are piecewise continuously-differentiable functions of quantities and supply is invertible from equilibrium prices; they showed the uniqueness of such equilibria. Holmberg [2009] restricted attention to symmetric equilibria in which bid functions are twice differentiable, and—assuming that the maximum supply strictly exceeds the maximum total quantity the bidders are willing to buy—proved the uniqueness of such smooth and symmetric equilibria.\(^\text{19}\) Ewerhart et al. [2010] and Ausubel et al. [2014] independently expand these analyses to Pareto supply with bounded support and linear marginal values. Restricting attention to equilibria in which bids are linear functions of quantities, they showed the uniqueness of such linear equilibria. In contrast, we look at all Bayesian Nash equilibria of our model, we impose no parametric assumptions (not even continuity) and we do not require that some part of the supply is not wanted by any bidder.\(^\text{20}\)

\(^{15}\)In our design analysis, we show that transparency is optimal for the pay-as-bid auctioneer. Thus optimal pay-as-bid auctions always admit a unique equilibrium, which is in pure strategies.

\(^{16}\)We establish the uniqueness of bids for relevant quantities—that is, for quantities a bidder wins with positive probability. Bids for quantities never obtained play no role in equilibrium outcomes. Our uniqueness result does not apply to these irrelevant bids.

\(^{17}\)Maximum likelihood-based estimation strategies (e.g. Donald and Paarsch [1992]) also rely on agents playing comparable equilibria across auctions in the data. Chapman et al. [2005] discuss the requirement of comparability of data across auctions.

\(^{18}\)See also, in a related context, Cantillon and Pesendorfer [2006].

\(^{19}\)Holmberg’s assumption that bidders do not want to buy part of the supply represents a physical constraint in the reverse pay-as-bid electricity auction he studies: in his paper bidders supply electricity and face capacity constraints—beyond a certain level they cannot produce more. This low-capacity assumption drives the analysis and it precludes directly applying the same model in the context of securities auctions in which bidders are always willing to buy more (provided the price is sufficiently low).

\(^{20}\)As a consequence of this generality, we need to develop a methodological approach which differs from that of the prior literature. McAdams [2002] and Ausubel et al. [2014] have also established the uniqueness
Our uniqueness result is also related to Klemperer and Meyer [1989] who established uniqueness in a duopoly model closely related to uniform-price auctions: when two symmetric and uninformed firms face random demand with unbounded support, then there is a unique equilibrium in their model.\textsuperscript{21} The main difference between the two papers is, of course, that Klemperer and Meyer analyze the uniform-price format, while we look at pay-as-bid.\textsuperscript{22}

Our bid representation theorem may be seen as a finite-market counterpart of Swinkels [2001], who studies pay-as-bid and uniform-price auctions in large markets, and in the limit, as the number of bidders goes to infinity, our representations are equivalent. He restricts attention equilibria that are asymptotically environmentally similar, an assumption we do not need. Our contribution also lies in establishing the representation of bids as averages of marginal values in all finite markets and not only in the limit. Holmberg [2009] derives a closed-form representation for symmetric and smooth equilibria subject to constraints on supply. We make no such assumptions, and instead prove that equilibria are symmetric and smooth; our results therefore provide support for his analysis and our finite-market representation of bids as weighted averages of marginal values is new.

Our bid representation result is surprising in the context of prior finite-market literature, which can be naturally read as suggesting that pay-as-bid equilibria are complex in the environments we study.\textsuperscript{23} Prior constructions of finite-market equilibria focused on the setting in which bidders’ marginal values are linear in quantity and the distribution of supply is (a special case of) the generalized Pareto distribution; see Wang and Zender [2002], Federico and Rahman [2003], Hästö and Holmberg [2006], Holmberg [2009], Ewerhart et al. [2010], and Ausubel et al. [2014]. This literature expressed equilibrium bids in terms of the intercept and slope of the linear demand and the parameters of the generalized Pareto distribution. Our treatment is not only more general but it also avoids the complexity inherent in expressing bids in terms of parameters of the functional forms studied in the earlier literature.

Our transparency result—that deterministic selling strategies are optimal—may appear of equilibrium in their respective parametric examples with two bidders and two goods.

\textsuperscript{21}The analogue of their unbounded support assumption is our assumption that the support of supply extends all the way to no supply. While the two assumptions look analogous they have very different practical implications. In a treasury auction, for example, a seller can guarantee that with some tiny probability the supply will be lower than the target; in fact, in practice the supply is often random and our support assumption is satisfied. On the other hand, it is substantially more difficult, and practically impossible, for the seller to guarantee the risk of arbitrarily-large supplies. Note also that we have known since Wilson [1979] that the uniform-price auction may admit multiple equilibria. No similar multiplicity constructions exist for pay-as-bid auctions.

\textsuperscript{22}The proof of our uniqueness result follows a differential analysis familiar from uniqueness results for first-price auctions (see, e.g., Maskin and Riley [2003], Lizzeri and Persico [2000], and Lebrun [2006]), but our analysis establishing the initial condition for the differential analysis is distinct.

\textsuperscript{23}We focus our discussion on settings with decreasing marginal utilities; for constant marginal utilities see Back and Zender [1993], and Ausubel et al. [2014] among others.
familiar from the no-haggling theorem of Riley and Zeckhauser [1983]. However, in multi-object settings the reverse has been shown by Pycia [2006] and Manelli and Vincent [2006]; and, as mentioned above, nondeterministic supply may have a role in uniform-price auctions. Furthermore, there is a subtlety specific to pay-as-bid that might suggest a role for randomization: by randomizing supply below the monopoly quantity, the seller forces bidders to compete and bid more for these quantities, and in pay-as-bid the seller collects the raised bids even when the realized supply is near the monopoly quantity. We show that, despite these considerations, committing to deterministic supply is indeed optimal.

We also establish a disclosure result: independent of the parameterization of the auction, the seller prefers to commit to announce the realization of supply prior to bid submission. Whether to reveal supply is an important question in treasury auctions, where the seller has pertinent information on supply prior to the auction. The reason for this disclosure result (as well as the preceding transparency result) is a novel bound on revenues in pay-as-bid auctions with random supply, rather than Milgrom and Weber’s [1982] celebrated linkage principle; the linkage principle is known to fail in the multi-unit auction context, cf. Perry and Reny [1999] and Vives [2010]. Furthermore, while our setting is one of Bayesian persuasion and information design, the full disclosure we establish stands in stark contrast to Kamenica and Gentzkow’s [2011] paradigmatic insight that in information design and Bayesian persuasion the sender wants to withhold—or obfuscate—information. Related to information design, Bergemann et al. [2017] and Bergemann et al. [2019] also find the optimality of withholding information in single-unit auctions. The reason why there is no contradiction between these results and our finding the optimality of full disclosure is that their sender possesses information about bidders’ valuations while in our analysis the bidders (receivers) are fully informed of their own value functions and the seller (sender) has and can release information about the quantity supplied, which is a key element of the bidders’ strategic interaction.

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24 We further establish that the seller sets deterministic supply in all perfect Bayesian equilibria of the game in which the seller first designs a pay-as-bid auction and then the bidders bid; this is in contrast to uniform-price auction design games for which we construct equilibria in which the seller sets random supply (even though we show that the revenue-maximal equilibrium has deterministic supply in the uniform-price auction, as in the pay-as-bid auction). Note that Chen et al. [2019] show that individual outcomes of a given random mechanism can be replicated by a deterministic mechanism when there are multiple privately informed participants, while we show that not only can the maximal revenue generated by any random pay-as-bid auction be obtained by some deterministic mechanism, but also that this is possible without fundamentally changing the auction mechanism.

25 The empirical impact of transparency has been extensively studied in the context of over-the-counter markets, for a recent review of this literature see e.g. Garratt et al. [2019]. The impact of transparency in uniform-price auctions has been experimentally studied by Hefti et al. [2019].

26 In single-unit auctions bidders necessarily have full information regarding the quantity supplied, and the auctioneer’s role in information design is inherently limited. Fang and Parreiras [2003] and Board [2009]
mechanism design (cf., e.g., Pavan et al. [2014]).

While we are not aware of prior literature on optimal design of supply and reserve prices in pay-as-bid, the general mechanism design question was addressed by Maskin and Riley [1989]: what is the revenue-maximizing mechanism to sell divisible goods? The optimal mechanism they described is complex and in practice the choice seems to be between the much simpler auction mechanisms: pay-as-bid and uniform-price. On the other hand, design issues have been addressed in the context of uniform-price auction. The design analysis of uniform-price focused on preventing collusive equilibria: Fabra [2003] and Marszalec et al. [2020] show that collusion is easier in uniform price than in pay as bid; however Klemperer and Meyer [1989] point out that the auctioneer can induce competition in a uniform-price auction by introducing slight randomness in supply, Kremer and Nyborg [2004] look at the role of tie-breaking rules, LiCalzi and Pavan [2005] and Burkett and Woodward [2020b] at elastic supply, McAdams [2007] at commitment, and Burkett and Woodward [2020a] at the role of price selection. By proving equilibrium uniqueness for pay-as-bid we show its resilience to equilibrium collusion, thus providing a pay-as-bid counterpart for this literature. We also contribute to this uniform-price literature directly by showing that not only the seller but also the bidders might be made worse off by the possibility of tacit collusion; the reason is that the seller who expects a collusive equilibrium in uniform-price auction might optimally respond by setting a high reserve price, thus recovering some of the revenue at the cost of bidders’ surplus.

Our revenue and welfare comparisons between pay-as-bid and uniform-price auctions contribute to the rich discussion of the pros and cons of these two formats. Swinkels [2001] focused on equilibria satisfying an asymptotic environmental similarity assumption and showed that pay-as-bid and uniform-price are revenue- and welfare-equivalent in large markets; Jackson and Kremer [2006] find revenue- and welfare-equivalence in large market limit under study the limits of the linkage principle and the resulting benefits of information withdrawal or obfuscation. The optimality of obfuscation generally obtains in setting in which the participation constraints are interim and the seller cannot charge for information (cf. Bergemann and Pesendorfer [2007]). Even if the seller can charge for information, obfuscation is shown to be optimal by Li and Shi [2017] except under orthogonality assumptions of Eső and Szentes [2007]. Obfuscation is also established in other settings in which—like in our auction setting—the sender’s interest (more revenue) is fundamentally misaligned with the bidders’ interests (reducing payment); in a global games context see, e.g., Li et al. [2020]. For analysis of bidders’ investment in information acquisition in auctions see e.g. Persico [2000] who finds that bidders in first-price auctions acquire more value-relevant information than bidders in second-price auctions. Finally, while we study a seller/sender who is able to commit to a disclosure strategy, our disclosure result immediately implies that a sender unable to commit would also fully reveal supply information. For information disclosure under no commitment see e.g. Grossman and Hart [1980] and Milgrom [1981].

Furthermore, in the environments we focus on, the bidders’ private information is correlated and hence the seller can nearly extract their full surplus using Crémer-McLean-type mechanisms [Crémer and McLean, 1988]; cf. footnote ??.
the assumption that the proportion of supply to the number of bidders vanishes to zero; our equivalence result does not rely on the size of the market, nor on an environmental similarity assumption, nor on extreme competition among bidders. Wang and Zender [2002] find pay-as-bid revenue superior in the equilibria of the complete-information linear-Pareto model they consider, and Woodward [2021] extends this dominance to mixed-price combinations of pay-as-bid and uniform-price auctions. Ausubel et al. [2014] show that—with ex-ante asymmetric bidders with flat demands—their format can be revenue superior. Our results on approximate revenue equivalence with small informational asymmetries complement this ambiguity: for uniform-price to raise significantly more revenue than pay-as-bid, bidders must be significantly asymmetric. In aggregate, prior theoretical work on the pay-as-bid versus uniform-price question has focused on revenue comparisons for fixed supply distributions and has allowed for neither reserve price nor supply optimization; indeed, the previous studies of pay-as-bid auctions with decreasing marginal values employed parametric specifications that did not support the analysis of design questions. Thus these results cannot address whether a well-designed pay-as-bid auction is preferable to a well-designed uniform-price auction. We go beyond these earlier papers both by allowing for the seller’s optimization and by imposing no assumptions on the seller’s information about the bidders.

Our divisible-good optimal revenue equivalence result provides a benchmark for the long-standing empirical debate whether pay-as-bid or uniform-price auctions raise higher expected revenues. This debate has attracted substantial empirical attention, with Hortaçsu and McAdams [2010] and Barbosa et al. [2020] finding no statistically significant differences in revenues, Février et al. [2002], Kang and Puller [2008], Armantier and Lafhel [2009], Marszalec [2017], Mariño and Marszalec [2020], and Hattori and Takahashi [2022] finding slightly higher revenues in pay-as-bid, and Goldreich [2007], Castellanos and Oviedo [2008], Armantier and Sbaï [2006], and Armantier and Sbaï [2009] finding slightly higher revenues in uniform-price. Hortaçsu et al. [2018] argue that the revenues are similar.

When bidders have symmetric or non-flat demands, pay-as-bid is revenue superior in all examples they consider. The special supply distributions these papers consider are not revenue-maximizing, hence there is no conflict between their strict rankings and our revenue equivalence. See also Jackson and Kremer [2006] and Fabra et al. [2006] who find that—with non-optimized supply—either format can be revenue superior, and Anwar [1999] and Engelbrecht-Wiggans and Kahn [2002] for revenue comparisons with flat demands. Fabra et al. [2011] show that the two formats may lead to the same investments in capacity. Hinz [2004] shows revenue equivalence in multi-unit auctions with single-unit demand.

They note that bids in U.S. Treasury auctions are typically “flat” and infer that not much surplus is retained by bidders; an alternative explanation highlighted by our analysis is that the bidders are close to being completely informed. Note also that while flatness implies that there is not much difference between the revenues generated by the pay-as-bid and uniform-price auctions, the uniform-price auction brings large downside potential in the form of collusive-seeming behavior. Our uniqueness theorem shows that the pay-as-bid format mitigates this risk. Of note in this context is Häfner [2020], who empirically demonstrates that bidders overbid in pay-as-bid auctions.
above, several of these papers conduct a counterfactual estimation of uniform-price revenues assuming truthful bidding, which is precisely the equilibrium selection under which our theoretical revenue and welfare equivalence obtains.\footnote{We show that revenue is maximized in the uniform price auction when the seller offers a deterministic quantity for sale, and bidders bid their true marginal values for the quantity they obtain. Whether they bid their true marginal values for quantities they do not obtain is irrelevant in equilibrium, thus we refer to truthful bidding at the received quantity as truthful bidding. This assumption is satisfied in some counterfactual approaches, where a bid curve is truthful if it matches the bidder’s marginal value curve.}

Our results regarding the selection of auction format have other empirical implications. We show in our analysis of the auction design game that the auctioneer either strictly prefers a pay-as-bid auction or is indifferent between the pay-as-bid and uniform-price formats. All else equal, our model suggests that pay-as-bid auctions should be more prevalent than uniform-price auctions. This claim is supported by the multi-country analyses of security auction implementation in Brenner et al. [2009] and OECD [2021], which find that pay-as-bid auctions are implemented by more than twice as many nations as implement uniform-price auctions, as well as the analysis of electricity markets in Del Río [2017], which finds that pay-as-bid auctions represent nearly 90% of auctions for allocating the capacity for renewable electricity generation.\footnote{Among electricity markets surveyed by Maurer and Barroso [2011], about half use pay-as-bid, and most of the remaining markets use uniform price.} Additionally, counterfactual analysis of uniform-price auctions assumes truthful reporting of values to obtain an upper bound on unobserved revenue. Our results can be taken to show that this bound is tight only if bidders are playing a seller-optimal equilibrium; otherwise there may be a significant divergence between observed revenue and counterfactual predictions. Furthermore, since revenue-dominance of the pay-as-bid auction implies that a seller should implement the uniform-price format only if she expects this favorable equilibrium to be played, we should expect counterfactual analysis from witnessed uniform-price auctions to find approximate revenue equivalence.

In our supplementary note [Pycia and Woodward, 2020] we provide additional results and applications that complement the present paper. We also extend the sharp results of this paper to approximate results in the case where there is asymmetric information among bidders. Complementing the equilibrium uniqueness for pay-as-bid, in the supplementary note we analyze equilibrium multiplicity in uniform-price auctions. We also apply our transparency result to study the relationship of the pay-as-bid auctioneer to a classical monopolist. We show that the auctioneer’s design problem is separable, and that the decisions of optimal supply and optimal reserve may be analyzed independently and show that this implies that increased variance of bidder values is good for the seller.

Finally, let us note that our analysis of pay-as-bid auctions can be reinterpreted as a model of dynamic oligopolistic competition among sellers who at each moment of time compete à
la Bertrand for sales and who are uncertain how many more buyers are yet to arrive. Prior sales determine the production costs for subsequent sales, thus the sellers need to balance current profits with the change in production costs in the future. This methodological link between pay-as-bid auctions and dynamic oligopolistic competition is new, and we develop it in follow up work.\footnote{The oligopolistic sellers uncertain of future demands correspond to bidders in the pay-as-bid auction, and sellers’ costs correspond to bidders’ values. For prior studies of dynamic competition see e.g. Deneckere and Peck [2012]; while they study competition among a continuum of sellers, the pay-as-bid-based approach allows for the strategic interaction between a finite number of sellers. The other canonical multi-unit auction format, the uniform-price auction, was earlier interpreted in terms of static oligopolistic competition by Klemperer and Meyer [1989].}

## 2 Model

There are $n \geq 2$ bidders, $i \in \{1, \ldots, n\}$. Bidder $i$’s marginal valuation for quantity $q$ is denoted $v(q; s)$, where $s$ is a signal known by all bidders but not by the seller.\footnote{In the 2020 draft of this paper, we showed that our results hold approximately when private information is allowed and that Theorem 1 does not depend on the ex ante symmetry of the bidders.} The seller believes that the signal comes from some distribution $\sigma$. We assume that $v(q; s)$ is strictly decreasing in $q$ where it is strictly positive, Lipschitz continuous, and almost-everywhere differentiable in $q$. We allow arbitrary dimensionality of $s$, and an arbitrary integrable $v(q; \cdot)$. The common signal $s$ has no strategic importance for bidders participating in an auction, and thus when studying the equilibrium among such bidders in Section 3, we fix $s$ and denote the bidders’ marginal valuation by $v(q; s) = v(q)$. Bidders’ information plays an important role in the analysis of the seller’s problem in Sections 4 and 5. The seller may not know the bidders’ information if, for example, the seller needs to commit to the auction mechanism before this information is revealed. Alternatively, the seller may want to fix a single design for multiple auctions.

To simplify the exposition of the design problem, we normalize the seller’s cost to 0. Our insights do not hinge on this normalization, and remain valid for any convex increasing cost function.\footnote{The reason why more general cost functions do not substantively change the analysis is that the Transparency Theorem (Theorem 6), on which the analysis of design builds, is valid irrespective of seller’s cost function. We provide more detailed discussion in Section 4.} Our design analysis builds on the existence, uniqueness, and bid representation results for pure-strategy Bayesian Nash equilibria of the pay-as-bid auction. We thus start by analyzing such equilibria. In our equilibrium analysis we assume that aggregate supply $Q$ is drawn from distribution $F$ with support $[0, Q]$, and we further assume that $F$ is Lebesgue absolutely continuous on $(0, Q)$ with continuous density $f > 0$; we also allow $F$ with full
mass concentrated at one point. $Q$ is independent of the bidders’ signal $s$.\footnote{The pure-strategy, support, and independence assumptions are relaxed in Appendix A. $Q$ might be an on-path or off-path supply in seller’s design problem or it might represent e.g. supply net of non-competitive bids; c.f. the discussion of Corollary 5.} Otherwise we impose no global assumptions on $F$.

In our analysis of auction design, the seller is free to choose any distribution $F$ satisfying the above conditions, as long as $\bar{Q} \leq Q^{\text{max}}$, where $Q^{\text{max}}$ is the maximum supply available to the seller.\footnote{We could allow for infinite $Q^{\text{max}}$ as long as the optimal monopoly quantity remains finite. This would be so if, e.g., the seller faces increasing and convex marginal costs of supply, or there is no heavy tail of marginal values.} The seller also implements a reserve price $R \geq 0$. While reducing the supply $Q$ and setting the reserve price $R$ play similar roles in the seller’s design problem, in Section 4.3, we show that both of these instruments are needed to maximize the revenues when the environment is sufficiently rich.\footnote{The relative virtues of regulating prices versus quantities have been studied since Weitzman [1974]. The potential benefit of a hybrid system regulating both prices and quantities was first studied by Roberts and Spence [1976].} When the seller employs both of these instruments, the quantity that is allocated is equal to $Q$ if the reserve is not binding, but it may be lower than $Q$ when the reserve price is binding. For any realized quantity $Q \leq \bar{Q}$ and bidders’ signal $s$, denote $Q^R (Q, s) = Q$ for reserve price $R = 0$ and $Q^R (Q, s) = \min \{Q, \sum_{i=1}^{n} v^{-1} (R; s)\}$ for $R \in (0, v(0, s)]$, where $v^{-1} (\cdot; s)$ is the inverse function of value given bidders’ signal $s$ (the inverse is well defined for $R \in (0, v(0, s)]$). Our Theorem 1 below implies that $Q^R (Q, s)$ is the quantity that is actually allocated. In particular, when the reserve is binding, the theorem implies that each bidder receives quantity $Q^R (Q, s) / n = v^{-1} (R; s)$. We use $\overline{Q}^R = Q^R(\bar{Q}, s)$ to denote the effective quantity at the maximum supply $\bar{Q}$.

In the pay-as-bid auction, each bidder submits a weakly decreasing bid function $b^i(q) : [0, \overline{Q}] \to \mathbb{R}_+$. Without loss of generality we may assume that the bid functions are right-continuous.\footnote{This assumption is without loss because we study a perfectly-divisible good and we ration quantities pro-rata on the margin. Indeed, we could alternatively consider an equilibrium in strategies that are not necessarily right-continuous. By assumption, the equilibrium bid function of a bidder is weakly decreasing, hence by changing it on measure zero of quantities we can assure the bid function is right continuous. Such a change has no impact on this bidder’s profit, or on the profits of any of the other bidders, because rationing pro-rata on the margin is monotonic in the sense of footnote 39. In fact, there is no impact on bidders’ profits even conditional on any realization of $Q$.} The auctioneer then sets the market-clearing price (also known as the stop-out price)

$$p^* = \max \left\{ R, \sup \{p' : q_1 + \ldots + q_n \geq Q \text{ for all } q_1, \ldots, q_n \text{ such that } b^1(q_1), \ldots, b^n(q_n) \leq p'\} \right\}.$$ 

If the set over which the supremum is taken is empty, then the stop-out price is set to the reserve price $R$. Agents are awarded a quantity associated with their demand at the stop-out price.
price,
\[ q_i = \max \left\{ q' : b^i (q') \geq p^* \right\}, \]
as long as there is no need to ration them. When necessary, we ration pro-rata on the margin, the standard tie-breaking rule in divisible-good auctions. The details of the rationing rule have no impact on the analysis of equilibrium bidding we pursue in Section 3.\(^{39}\) The demand function (the mapping from \( p \) to \( q^i \)) is denoted by \( \varphi^i (\cdot) \).\(^{40}\) Agents pay their bid for each unit received, and utility is quasilinear in monetary transfers; hence,
\[ u^i (b^i) = \int_0^{q^i (p^*)} v (x) - b^i (x) \, dx. \]

## 3 Pay-as-Bid Equilibrium

In this section we analyze pure strategy equilibria. In the analysis we hold bidders’ signal \( s \) fixed, and denote the bidders’ symmetric marginal valuation by \( v^i (q; s_i) = v(q) \). We begin the analysis by providing a tight bound on the market price, then we leverage this bound to provide a closed-form expression for the unique equilibrium bid profile.

### 3.1 Minimum Market Price

Our analysis of optimal bidding relies on the following key theorem proven in Supplementary Appendix B; in this theorem we allow mixed-strategy equilibria.\(^{41}\)

**Theorem 1.** [Minimum Market Price] In any mixed-strategy equilibrium of the pay-as-bid auction, for any signal \( s \), the market clearing price for the effective maximum quantity \( Q^R \) is given by
\[ p \left( Q^R ; s \right) = v \left( \frac{1}{n} Q^R ; s \right). \]

As we allow mixed strategies, \( p \left( Q^R ; s \right) \) could in principle be random; part of the theorem’s claim is that it is deterministic. The equality of the market price at the maximum supply and each bidder’s marginal value at the last unit they receive is illustrated in Figure

\(^{39}\)The only place when we rely on rationing rule is the analysis of reserve prices but even in this part of the analysis all we need is that rationing rule is monotonic: that is, the quantity assigned to each bidder increases when the stop-out price decreases; rationing pro-rata on the margin satisfies this property.

\(^{40}\)Where \( b^i (\cdot) \) is constant, \( \varphi^i \) is not well-defined. Where important, we will use \( \varphi^i \) and \( \overline{\varphi}^i \) to denote the right- and left-continuous (respectively) inverses of \( b^i \), \( \varphi^i (p) = \sup \left\{ q : b^i (q) > p \right\} \) and \( \overline{\varphi}^i (p) = \sup \left\{ q : b^i (q) \geq p \right\} \).

\(^{41}\)To avoid trivialities, in the formulation of the theorem we restrict attention to mixed strategies for which every action in the support is a best response. This restriction has no substantive impact on anything in the paper except that, were we to allow mixed strategies that put probability \( 0 \) on non-best response actions, the characterization of price in Theorem 1 would obtain with probability \( 1 \), with no change in the proof.
Figure 1: In equilibrium, bids must equal values at the maximum quantity which can be received (Theorem 1). Otherwise, a small upward deviation can obtain a discretely greater quantity at minimal additional cost.

2. Theorem 1 determines the minimum market price because the market price is weakly decreasing in total quantity sold (an implication of bids being weakly decreasing in quantity), and hence the market price is minimized at effective maximum supply $Q^R$. The market-clearing price at supply lower than $Q^R$ can (and frequently does) rise above the lower bound $v(Q^R/n; s)$.

The intuition for this theorem is that a bidder with a strictly positive margin at the maximum feasible quantity could slightly increase their bid and obtain a non-negligible additional quantity at minimally higher price, which would be a profitable deviation. The proof of Theorem 1 formalizes this intuition and takes care of technical complications related to tie-breaking, flat bids, and binding monotonicity constraints. Of course, this intuition applies only to the maximum quantity at which the increased bid is paid only when it is marginal; at any lower quantity the increased bid would need to be paid also when inframarginal.

Theorem 1 plays a crucial role in the equilibrium uniqueness result for symmetrically informed bidders we turn to next, and therefore in many of our subsequent results.

3.2 Existence, Uniqueness, and Bid Representation

We first show that equilibrium is unique and tractable whenever it exists. The existence of equilibrium can then be analyzed in terms of what equilibrium strategies must be, if an equilibrium exists. We therefore defer discussion of existence until after our uniqueness and representation results, and for expositional simplicity our uniqueness and representation results are formulated conditional on the existence of Bayesian Nash equilibrium. Proofs of all results may be found in Supplementary Appendix C.
We focus on relevant quantities, by which we mean the quantities that a bidder can win with positive probability in equilibrium. We say that an equilibrium is essentially unique if the set of relevant quantities and the bids on relevant quantities are the same in all equilibria; in particular, the market-clearing price, payments, and allocations conditional on the realization of supply is then the same in all equilibria; bids for quantities which the bidder never receives do not need to be uniquely determined.

**Theorem 2. [Uniqueness]** The Bayesian Nash equilibrium is essentially unique.

To get a sense why this theorem obtains, note that if we restricted attention to symmetric and smooth equilibria satisfying the first order condition (which we do not), then uniqueness would follow from Theorem 1. Indeed, in a symmetric smooth equilibrium bidders’ first-order conditions give us an ordinary differential equation and Theorem 1 provides us with a unique initial condition for this equation by uniquely determining the price \( p(\overline{Q}^R) \) at the maximum supply and hence, in a symmetric equilibrium, the bids for quantity \( \overline{Q}^R/n \). The proof, provided in Supplementary Appendix C, builds on this idea and addresses the difficulties raised by potential asymmetries, non-differentiabilities, and discontinuities.\(^{42}\)

Our analysis of uniqueness allows us to construct equilibrium bidding strategies, which turn out to be surprisingly tractable. We formulate the strategies using the auxiliary concept of a *weighting distribution* (discussed after the theorem): for any quantity \( Q \in [0, \overline{Q}) \), the \( n \)-bidder weighting distribution has c.d.f.

\[
F^{Q,n}(x) = 1 - \left( \frac{1 - F(x)}{1 - F(\overline{Q})} \right)^{n-1};
\]

note that \( F^{Q,n} \) has support \([Q, \overline{Q}]\) and increases from 0 when \( x = Q \) to 1 when \( x = \overline{Q} \).

**Theorem 3. [Bid Representation]** The essentially unique equilibrium is symmetric. For any quantity \( q \in [0, \overline{Q}^R/n] \), the bid \( b^i \) of each bidder \( i \) is given by

\[
b^i(q) = \int_{mq}^{\overline{Q}} v \left( \frac{\min\{x, \overline{Q}^R\}}{n} \right) dF^{mq,n}(x). \tag{1}\]

\(^{42}\)Our uniqueness result stands in contrast to nonuniqueness results in uniform-price auctions (cf. Klemperer and Meyer [1989]) and in Bertrand competition with convex costs (cf. Weibull [2006]). We discuss uniform-price auctions in Section 5 (cf. Lemma 1). In Bertrand competition, convex costs correspond to our decreasing marginal value curve. We obtain uniqueness where Bertrand competition allows nonuniqueness because our bidders’ strategy space is larger. In particular, Bertrand competitors who undercut must supply all market demand whether or not doing so is profitable, while our bidders may submit a limit bid which yields them only as much quantity as they desire. For a discussion of uniqueness in Bertrand competition see e.g. Burguet and Sákovics [2017].
We impose no assumptions on symmetry of equilibrium bids, their strict monotonicity, nor continuity or differentiability; we derive all these properties. Because the unique equilibrium is symmetric, the bid functions allow us to express the market price for any realization of supply \( Q \in [0, \overline{Q}] \) as

\[
p(Q) = b^i \left( \frac{Q}{n} \right) = \int_Q^\overline{Q} v \left( \frac{\min \{ x, \overline{Q}^R \} }{n} \right) d F^{Q,n}(x). \tag{2}
\]

Furthermore, when the reserve price does not bind, formulas (1) and (2) simplify, as \( \overline{Q}^R = \overline{Q} \) and \( \min \{ x, \overline{Q}^R \} = x \); in this case the equilibrium bid equation can be rewritten as

\[
b^i(q) = \int_{\min \{ q, \overline{Q}^R \}}^{\overline{Q}^R} v \left( \frac{x}{n} \right) d F^{nq,n}(x) + v \left( \frac{\overline{Q}^R}{n} \right) \left( 1 - F^{nq,n}(\overline{Q}^R) \right).
\]

When the reserve price is binding, \( R > v(\overline{Q}) \), the bid function is the same as if the supply was distributed on \( [0, \overline{Q}^R] \) with a mass point at \( \overline{Q}^R \).

The weighting distributions depend only the number of bidders and the distribution of supply, and not on any bidder’s true demand. As the number of bidders increases the weighting distributions put more weight on lower quantities. In the limit, on its support \( F^{Q,n}(x) \) converges to \( \frac{F(x) - F(Q)}{1 - F(Q)} \) that is to the distribution of supply conditional on it being above \( Q \). The density \( d F^{Q,n}(x) / dx \) then approaches \( f(x) / (1 - F(Q)) \), the conditional density at \( x \) given that realized supply is at least \( Q \). Thus, in the limit, the theorem expresses the bid for quantity \( q \) as the average marginal value for the marginal unit, conditional on receiving quantity above \( q \). In other words, in the limit the bid on any relevant quantity \( q \) is equal to the expected market clearing price conditional on the bidder receiving \( q \), that is conditional on the event that lowering the bid would have affected the ex post payoff of the bidder; a corresponding limit economy result is established in Swinkels [2001]. In the competitive limit the bidder bids away all marginal rents. Expected utility is still positive since marginal utility is decreasing in quantity, hence bidding away marginal rents leaves rents for inframarginal units.

Away from the competitive limit, the bidder also retains rents on marginal units. The fewer the bidders, the more market power the bidders have and the higher are their rents on marginal units: this is reflected in the exponent \( (n - 1)/n \) in the weighting distribution \( F^{Q,n} \). The equilibrium bids \( b^i \) are appropriately-weighted averages of bidders’ marginal values \( v \), and in this they resemble both the bids in the competitive limit and the bids in first-price auctions with privately-informed bidders. Because marginal values are decreasing
in quantity, bids are below values—that is, bidders are shading their bids—except for the bid on the effective maximum quantity where limit equality obtains, an equality consistent with Theorem 1.

In the special case when supply is deterministic, our bid representation implies that the bid function is flat on quantities up to $\overline{Q}^R/n$. It can be easily seen that flat bids can be supported in an equilibrium. Given deterministic supply the bidders know exactly the quantities they will receive in equilibrium: a deviation increasing the bid for lower quantities increases the payment to the seller without improving the bidder’s allocation; a deviation decreasing the bid decreases the allocation and the decrease discourages the deviation provided opponents’ bids on quantities above $\overline{Q}^R/n$ are sufficiently high. The argument for random supply is in the Appendix.

As an example note that when marginal values $v$ are linear and the supply distribution $F$ is generalized Pareto, $F(x) = 1 - (1 - x^{\frac{1}{\alpha}})^{\alpha}$ for some $\alpha > 0$, then our bid representation shows that the equilibrium bids are linear in quantity. The linear-Pareto case of our general setting has been analyzed by Ewerhart et al. [2010], and Ausubel et al. [2014], who constructed the linear equilibrium directly in terms of the slope and incident of demand and the parameters of the Pareto distribution. Our general results contribute to our understanding of this example by allowing us to conclude that the linear equilibrium is essentially unique in the class of all pure-strategy equilibria, and that bids remain linear in the linear-Pareto setting even in the presence of a reserve price. Our results are of course more broadly applicable and Figure 2 illustrates equilibrium bids for ten bidders with linear marginal values who face a distribution of supply that is truncated normal.\footnote{In all figures, we check our equilibrium existence condition and calculate bids numerically using R. In Figure 2 we use a normal distribution with mean 3 and standard deviation 1, truncated to the interval [0,6].}

Our bid representation theorem allows us to establish when an equilibrium exists because it derives the unique equilibrium bids on relevant quantities, conditional on equilibrium existence. When these bids are played in an equilibrium, we can express the expected utility of a bidder $i$ as

$$\mathbb{E}u^i = \int_0^\overline{Q} U(q; q) dq,$$

where $U : [0, \overline{Q}]^2 \to \mathbb{R}$ is given by

$$U(\hat{q}; q) = (v(q) - b(\hat{q})) (1 - F(q + (n - 1) \hat{q})),$$

and $b$ is the symmetric bid function derived in Theorem 3.

**Theorem 4. [Existence]** There exists a pure-strategy Bayesian Nash equilibrium in the pay-as-bid auction whenever, for almost every $q \in [0, \overline{Q}/n]$, the first derivative of $U(\cdot; q)$ is
Figure 2: Equilibrium bids when values are linear and the distribution of supply \( Q \) is truncated normal. This and the subsequent figures represent bids, marginal values, and the c.d.f. of supply; to identify the three curves note that bids and the marginal values are decreasing, bids are below marginal values, and the c.d.f. is increasing.

zero only at the global maxima of \( U(\cdot; q) \).

The proof of this theorem in Appendix C extends the bidding strategies \( b^i(q) = b(q) \) from Theorem 3 beyond relevant quantities \( q \). Assuming that bidders other than \( i \) play these strategies, we establish that \( \int_0^{\overline{Q}/n} \max_{\hat{q} \in [0, \overline{Q}/n]} U(\hat{q}; q) dq \) is an upper bound on the bidder \( i \)'s expected utility irrespective of \( i \)'s bidding strategy. By showing that the first derivative of \( U(\cdot; q) \) is zero at \( \hat{q} = q \), we infer from the assumption of Theorem 4 that \( U(q; q) \) is a global maximum of \( U(\cdot; q) \), which allows us to conclude that bidder \( i \) maximizes his payoff (and achieves the above-mentioned upper bound) by following the strategy we consider; thus no bidder has a profitable deviation and the considered profile of strategies constitutes an equilibrium. Because Theorem 3 expresses the bid function \( b \) in terms of model fundamentals, Theorem 4 is not conditioning equilibrium existence on an equilibrium object.

Our sufficient condition is satisfied when, for example, the function \( U(\cdot; q) \) is pseudo-concave, and hence also when \( U(\cdot; q) \) is concave. Additionally, our sufficient condition is closed with respect to several changes of the environment: adding a bidder, making marginal values less concave (or more convex), and raising the reserve price all preserve existence. In regular problems, the existence condition is satisfied as soon as there sufficiently many bidders, which gives us the following:

Corollary 1. [Existence with many bidders] Suppose marginal values are differentiable and have slope bounded away from zero, and the density of per-capita supply is bounded away
from 0 on \((0, \bar{Q})\) and has bounded derivative. If there are sufficiently many bidders, then a pure-strategy Bayesian Nash equilibrium exists.

Regardless of market size, our sufficient condition is satisfied in the aforementioned linear-Pareto environment and it is satisfied whenever the inverse hazard rate \(H\) is increasing—hence when the hazard rate is decreasing—irrespective of the marginal value function \(v\).\(^{44}\) Our existence condition is satisfied in the examples illustrated in Figures 2-4, which include truncated normal distribution, strictly concave marginal values, and reserve prices.

Our existence condition is also satisfied when supply is deterministic. Suppose that the seller commits to supply quantity \(\bar{Q}\). As supply is deterministic, the auxiliary density \(dF_{Q,n}(x)\) is equal to 0 for all \(x < \bar{Q}\), and equilibrium bids are flat; the expression \((v(q) - b(\hat{q}))(1 - F(q + (n - 1)\hat{q}) = U(\hat{q}; q)\) is therefore constant on \(\hat{q} \in [0, \bar{Q}^R/n]\). Recall that we already verified the existence of equilibrium in the deterministic case in our discussion of Theorem 3.

While our sufficient condition shows that equilibrium exists in many cases of interest, there are situations in which the equilibrium does not exist; see our discussion in the introduction.

### 3.3 Equilibrium Properties and Comparative Statics

The bid representation of Theorem 3 has many implications. Among them are the comparative statics of this subsection and Section 4.3.

Our bid representation implies that supply concentration leads to flat bids and low margins on bids near the per-capita concentrated quantity. We say that a distribution is \(\delta\)-concentrated near quantity \(Q^*\) if \(1 - \delta\) of the mass of supply is within \(\delta\) of quantity \(Q^*\).

**Corollary 2. [Flat Bids]** For any \(\varepsilon > 0\) and quantity \(Q^*\) there exists \(\delta > 0\) such that, if supply is \(\delta\)-concentrated near \(Q^* \leq \bar{Q}^R\), then the equilibrium bids for all quantities lower than \(Q^* \leq \bar{Q}^R - \varepsilon\) are within \(\varepsilon\) of \(v\left(\frac{Q^*}{n}\right)\).

Figure 3 depicts the flattening of equilibrium bids predicted by Corollary 2; in the three sub-figures ten bidders face supply distributions that are increasingly concentrated around the total supply of 6 (per capita supply of 0.6). In the special case of deterministic supply, which is 0-concentrated, Corollary 2 implies that equilibrium bids are perfectly flat.

\(^{44}\)The existence of equilibrium in the linear-Pareto environment was established by Ewerhart et al. [2010] and Ausubel et al. [2014] for bounded generalized Pareto distributions and Wang and Zender [2002], Federico and Rahman [2003], and Holmberg [2009] for unbounded Pareto distributions. The sufficiency of decreasing hazard rate for equilibrium existence was established by Holmberg [2009]. Theorem 4 also implies the existence results of Swinkels [1999] and Jackson and Kremer [2006], who showed that an equilibrium exists in the limit as per-capita supply goes to zero.
Figure 3: Bids are flatter for more concentrated distributions of supply.

Remark 1. The practical implications of Corollary 2 may be observed in U.S. Treasury auctions for short-term securities. Hortaçsu et al. [2018] show that in these auctions supply randomness is low, and empirically-observed uniform-price bids are nearly flat. Because supply randomness is low, Corollary 2 implies that counterfactual pay-as-bid bids would also be nearly flat, and changing the auction format would yield little additional revenue.45

Corollary 3. [Low Margins] For any $\varepsilon > 0$ and quantity $Q^* \leq \bar{Q}^R$ there exists $\delta > 0$ such that, if supply is $\delta$-concentrated near $Q^*$, then each bidder’s equilibrium margin $v \left( \frac{1}{n} Q^* - \delta \right) - b \left( \frac{1}{n} Q^* - \delta \right)$ on the $\frac{1}{n} Q^* - \delta$ unit is lower than $\varepsilon$.

This corollary complements Theorem 1, which establishes that bidders obtain zero margin at the maximum quantity.

Our bid representation also implies that bidders’ equilibrium margins are lower and the seller’s revenue is larger when there are more bidders:

Corollary 4. [More Bidders and Market Place Mergers] Bidders submit higher bids and the seller’s revenue is larger and each bidder’s profits smaller when there are more bidders—both when the supply distribution is held constant, and when the per-capita supply distribution is held constant. In particular, the sum of revenues from markets with $n_1$ and $n_2$ bidders and the same per-capita supply distribution is less than the revenue from the joint market with $n_1 + n_2$ bidders.

The corollary follows because as the number of bidders increases, $1 - F^{Q,n}(x) = \left( \frac{1 - F(x)}{1 - F(Q)} \right) \frac{n-1}{n}$ decreases, and hence $F^{Q,n}(x)$ increases, thus mass in the weighting distribution is shifted towards lower $x$, where marginal values are higher. At the same time, the marginal value at

45Hortaçsu et al. [2018] use inferred marginal values to show that bidders do not obtain much surplus; thus changing in the auction format cannot yield much additional revenue. Our corollary goes beyond their analysis by showing that given flat uniform-price bids and relatively certain supply, changing the auction format also cannot cost much revenue.
Figure 4: Bids go up when more bidders arrive (and per capita quantity is kept constant) but not by much: 5 bidders on the left, 10 bidders in the middle, and 5 million bidders on the right. Note that all axis scales are identical.

$x$ either increases in $n$ (if we keep the distribution of supply constant) or stays constant (if we keep the distribution of per-capita supply constant). Our bid representation also implies that when per capita supply is similar in both divided markets, merging the markets will improve total revenue; however, if the two markets have substantially different per capita supply, then merging them might decrease total revenue. Similar market-merger conclusions have been derived for uniform-price auctions, cf., e.g., Rostek and Yoon [2021], Fabra and Llobet [2021], Wittwer [2021].

While bidders raise their bids when facing more bidders even if the per-capita distribution stays constant, our bid representation theorem implies that the changes are small.\footnote{In the limit, and absent a reserve price, bids take the simple form $b(q) = \int_{\max \text{Supp} F}^{\text{Supp} F} v(x) \, dF(x) / (1 - F(q))$ where $F$ is distribution of per-capita supply. Notice that if we keep the supply distribution fixed while more and more bidders participate in the auction, then in the large market limit revenue converges to average supply times the value on the initial unit. See Swinkels [2001] for limit results with fixed per-capita supply and Jackson and Kremer [2006] for limit results with fixed supply.}

This is illustrated in Figure 4 in which increasing the number of bidders from 5 bidders to 10 bidders has only a small impact on the bids, as does the further increase from 10 bidders to 5 million bidders.

4 Designing Pay-as-Bid Auctions: Transparency and Disclosure

In this section we maintain the assumption that the pay-as-bid format is run and analyze the design of such auctions. We focus on the reserve price and the distribution of supply, the two natural elements of pay-as-bid auction that the seller can select; as in the previous
section, we restrict attention to pure-strategy equilibria.\footnote{As pay-as-bid is largely employed by central banks and governments, the efficiency of allocations may be an important concern and a reason a seller may want to ensure that a pure-strategy equilibrium is being played. The symmetry of equilibrium strategies we prove in Theorem 3 implies that in pure-strategy equilibrium the marginal value for any unit received is higher than the marginal value for any unit not received. In a pure-strategy equilibrium, there are thus no efficiency improving re-allocations of units among bidders; this property trivially fails in any mixed-strategy equilibrium that is not essentially in pure strategies.} In Appendix A, we relax these restrictions—we allow the seller to set any distribution over elastic supply and we consider the possibility that the bidders play a mixed-strategy equilibrium—and we show that our insights on transparency and disclosure remain valid.

As design decisions are taken from the seller’s perspective, our terminology in this and the subsequent sections now explicitly keeps track of the bidders’ information. We impose no assumptions on the distribution \( \sigma \) other than \( v(q; \cdot) \) being integrable.

### 4.1 Transparency

The key insight that underlies our design analysis is that—in contrast to typical multidimensional mechanism design problems discussed in the introduction—in an optimized pay-as-bid auction deterministic—and, hence, transparent—supply is optimal. Furthermore, if supply is exogenously random, then it is optimal for the seller set a deterministic supply cap; and, independent of whether a supply cap is feasible, it is optimal to announce the realized supply to the bidders prior to the auction.

First, suppose that the seller has some deterministic quantity \( \bar{Q} \) of the good; we relax this assumption below. For any fixed reserve price, we consider the problem of designing a supply distribution \( F \) that maximizes the seller’s revenue. The seller has the option to offer a stochastic distribution over multiple quantities, and it is plausible that such randomization could increase his expected revenue.\footnote{In a treasury auction, a seller may commit to randomize supply sold at auction by setting it as a total supply net of sales to non-competitive buyers; cf. the discussion below of this practice in US and Japanese treasury auctions.} For instance, offering quantities lower than the optimal monopoly quantity, \( Q^* \), results in a tradeoff: the seller sometimes sells less than \( Q^* \), with a direct and negative revenue impact, but when he sells quantity above \( Q^* \) he will receive higher payments due to the pay-as-bid nature of the auction. This tradeoff is illustrated in Figure 3, in which concentrating supply lowers the bids.\footnote{A priori such trade-offs can go either way; see Pycia [2006] and the introduction.}

We show that selling the deterministic supply \( Q^* \) is in fact revenue-maximizing across all pure-strategy equilibria; for this reason in the sequel we refer to \( Q^* \) as optimal supply. In Appendix A, we further show that under deterministic (and possibly elastic) supply all mixed-strategy equilibria are in pure-strategies and that the unique equilibrium under opti-
mal deterministic supply revenue-dominates any mixed-strategy equilibrium at any random supply.

**Theorem 5. [Transparency of Optimal Supply]** In pure-strategy equilibria, the seller’s revenue under non-deterministic supply is strictly lower than under optimal deterministic supply. Optimal deterministic supply is given by the solution to the monopolist’s problem when facing uncertain demand.

To grasp why the result holds observe that the seller collects infinitesimal revenue $p(Q)dQ$ whenever realized supply is in $[Q, Q + dQ]$. From Theorem 3, we have $p(Q) = \hat{v}(Q; s) + \int_Q^\infty \hat{v}_Q(x; s)(1 - F_{Q,n}(x))dx$. For each $x \in [Q, \overline{Q}]$ the right-hand integral is included in bids for all $Q' < x$, and in expectation can be bounded by $x\hat{v}_Q(x; s)(1 - F(x))$. Since $\hat{v}(Q; s)$ is included with probability $1 - F(Q)$, expected revenue is bounded above by the expectation of $\hat{v}(Q; s) + Q\hat{v}_Q(Q; s)$, which is the derivative of monopoly revenue. It follows that supplying the monopoly quantity is optimal.

As the following proof sketch indicates, Theorem 5 remains valid if the reserve price is arbitrary rather than optimized. The transparency result also remains valid for sellers who maximize profits equal to revenue net of costs, provided the marginal cost curve is weakly increasing.\(^{50}\) Such sellers optimally choose the deterministic quantity (or quantity cap) that maximizes the expected revenue minus cost rather than the quantity that maximizes the expected revenue. Taking the cost into account affects what quantity is optimal, but it does not change the result that optimal supply is deterministic.

**Remark 2.** Equilibrium multiplicity in uniform-price auctions implies that the optimality of transparent supply in pay-as-bid auctions does not extend to uniform-price auctions. The reason is that the bidding equilibrium may have an irregular dependence on the reserve price and supply distribution in the uniform-price auction. We discuss the issue in the ensuing analysis of the auction design game; cf. our discussion of Lemma 1.

To prove Theorem 5, we start with an arbitrary reserve price and supply distribution and the induced pure-strategy equilibrium bids. Holding equilibrium bids fixed, we use our bid representation from Theorem 3 to bound expected revenue by the standard monopoly revenue given the supply distribution.\(^{51}\) In effect we obtain the following bound on the expected revenue,

\(^{50}\)In the absence of the increasing marginal cost assumption, an analogue of Theorem 5 would need to be modified to take account of resulting ironing. See also our remark at the end of the proof of the theorem in Supplementary Appendix E.

\(^{51}\)This argument hinges on re-assigning the revenue across supply realizations; in particular, the actual revenue conditional on a supply realization is not necessarily bounded by the revenue the seller would obtain by setting the deterministic supply fixed at the conditioning supply realization.
where $\pi^F(Q; s)$ is the seller’s revenue when bidders’ signal is $s$, the realization of supply is $Q$, and bidders bid against the distribution of supply $F$, while $\pi^\delta(Q; s)$ is the seller’s revenue when bidders’ signal is $s$, the realization of supply is $Q$, and bidders bid against the distribution of supply that puts probability 1 on supply quantity $Q$.\footnote{In a working version of this paper, we provide a tighter bound on expected revenue, in which we average $\pi^\delta(Q; s)$ over the auxiliary distribution $J \equiv 1 - (1 - F)^{(n-1)/n}$. The bound presented in equation (3) is simpler to derive, and is sufficient for all our results.} Note that $\pi^\delta(Q; s)$ is a monopolist’s profit from selling quantity $Q$ to buyers with common signal $s$. This upper bound implies that the seller’s revenue is maximized when the seller sets the supply to be always equal to the revenue-maximizing deterministic supply. We provide the details of the proof in Supplementary Appendix E (bound (3) above restates inequality (14) in the proof).

The structure of the proof of Theorem 5 has two important implications. First, under the additional restriction that $Q \in \mathbb{R}$ is single-peaked in $Q$, the proof is applicable to environments in which the seller’s underlying supply is random and the seller can lower the supply but cannot increase it above the underlying supply realization. In this more general environment we assume that the distribution of underlying supply is exogenously given by $F$ with a compact support.\footnote{We can replace the assumption that the support of $F$ is compact with other assumptions that guarantee that the optimal solution exists, such as for instance that there is a finite $q > 0$ such that for all $s$, $v(q; s) = 0.$} Our proof then shows that the revenue maximizing-supply reduction by the seller reduces supply to $Q^\star$ whenever the exogenous supply is higher than $Q^\star$, and otherwise leaves the supply unchanged. As discussed following Theorem 4 and Corollary 2, when supply is deterministic bids are flat at level $v(Q/n; s)$.

As an application of our analysis, note that multi-unit auctioneers frequently obtain revenue not only from competitive bidders but also from noncompetitive bidders who pay a fixed price determined by the auction’s outcome. For example, in France [Agence France Tr\{i\}ésor, 2022], the Czech Republic [Ministry of Finance, 2016], and Korea [Ministry of Economics, 2021] noncompetitive bidders receive supply in addition to the supply that is auctioned to competitive bidders. When the price paid by noncompetitive bidders is monotone in the auction’s market-clearing price, our Theorem 5 remains valid.

**Corollary 5. [Transparency of Optimal Supply with Noncompetitive Demand]** If the seller sets the distribution of the supply in the auction and the noncompetitive bidders pay a per-unit price that is weakly increasing in the auction market-clearing price, then the sum of seller’s revenue from competitive and noncompetitive bidders is maximized by setting deterministic supply in the auction.
Corollary 5 is straightforward to prove. Denote the equilibrium market-clearing price, conditional on competitive supply $Q_c$ and bidder signal $s$, by $p^F(Q_c; s)$, and let $p_{nc}(p^\star)$ be the price paid by noncompetitive bidders as a function of the market-clearing price $p^\star$. Considering payments from both competitive and noncompetitive bidders, the seller maximizes $E\left[\pi^F(Q_c; s) + p_{nc} \cdot p^F(Q_c; s)Q_{nc}\right]$ over $F$. Inequality (3) provides an upper bound on competitive revenue, $E\left[\pi^F(Q_c; s)\right] \leq \int_{0}^{Q_R} E_s \left[\pi^\delta(x; s)\right] dF(x)$, and since bids are below values Theorem 1 implies that, given a realized competitive quantity $Q_c$, the equilibrium market-clearing price $p^F(Q_c; s)$ is lower than the market price $p^\delta(Q_c; s)$ when bidders with signal $s$ know that competitive supply is $Q_c$. Because $p_{nc}$ is monotone in the market-clearing price, $\int_{0}^{Q_R} E_{s,Q_{nc}} \left[\pi^\delta(x; s) + p_{nc} \cdot p^\delta(x; s)Q_{nc}\right] dF(x)$ is an upper bound on the seller’s revenue, and in turn is bounded from above by $\max_{Q \in [0, Q_R]} E_{s,Q_{nc}} \left[\pi^\delta(Q; s) + p_{nc} \cdot p^\delta(Q; s)Q_{nc}\right]$. The seller can achieve this latter upper bound by setting deterministic supply equal to $\arg \max_{Q \in [0, Q_R]} E_{s,Q_{nc}} \left[\pi^\delta(Q; s) + p_{nc} \cdot p^\delta(Q; s)Q_{nc}\right]$. Note that this same argument works when $p_{nc}$ is stochastic and has expectation increasing in the market-clearing price.\footnote{Corollary 5 remains valid if the seller observes noncompetitive demand $Q_{nc}$ prior to setting the distribution of auctioned supply, because we may let $F_{nc}$ be a degenerate distribution placing all probability on $Q_{nc}$.\footnote{We maintain the global assumption of this section that all induced auction equilibria are in pure strategies; in Appendix A we show that the full disclosure insight remains valid without this assumption.}}

### 4.2 Full Disclosure

As another application of our analysis let us note that the seller who does not fully control the auctioned-off supply would like to fully reveal the realized supply. For instance, in the United States [TreasuryDirect, 2022] and Japan [Hattori and Takahashi, 2022], the seller announces joint supply of debt to be sold in an auction and allocated to noncompetitive bidders, and the supply sold in an auction is then the residual supply after noncompetitive bidders’ demand is filled.

The seller thus finds transparency optimal both in the sense of setting a deterministic supply (or supply cap) and in the sense of revealing the seller’s information about supply. To formalize this full-disclosure insight we enrich our base model as follows. We assume that the distribution of supply is exogenously given and commonly known. Before learning the realization of supply, the seller can publicly commit to an auction design (reserve price and supply restriction) and a disclosure policy; a disclosure policy maps the realization of supply to a distribution of public announcements (messages) from an arbitrary space of messages.\footnote{We maintain the global assumption of this section that all induced auction equilibria are in pure strategies; in Appendix A we show that the full disclosure insight remains valid without this assumption.} After publicly committing to a disclosure policy and an auction design, the seller learns the realization of supply and announces the message prescribed by the disclosure policy. Then,
the bidders learn their value and bid in the auction.

**Theorem 6. [Optimality of Information Disclosure]** The seller’s expected revenue is maximized when the seller commits to fully reveal the realization of supply.

Before presenting a surprisingly simple argument deriving this theorem from our preceding results, let us observe that Theorem 6 remains valid even if the seller does not optimize the reserve price and supply cap in the auction and these parameters of the auction are arbitrarily set, with no change in the proof. In addition, because we prove Theorem 6 for the environment in which the seller can commit to a disclosure strategy, the same full disclosure insights a fortiori holds true for environments where the seller cannot commit.

**Proof.** Suppose that the seller commits to a disclosure strategy and this strategy leads to a message that induces the bidders to believe that the (conditional) distribution of supply is \( \hat{F} \) with upper bound of support \( \hat{Q} \). The revenue bound obtained in the proof of Theorem (5) gives

\[
E \left[ \pi^F (Q; s) \right] \leq \int_0^{\hat{Q}} \mathbb{E}_s \left[ \pi^\delta (x; s) \right] d\hat{F} (x),
\]

and thus expected revenue is bounded above by the expected revenue obtained by the seller fully revealing to the bidders the realization of supply. In consequence, the seller’s expected revenue is maximized when the seller ex ante commits to fully reveal the realization of supply.

While the seller in Theorem 6 maximizes auction revenue, an analogous result obtains if the seller maximizes the total revenue obtained from not only from the allocation to competitive bidders who submit demand curves, but also from the noncompetitive bidders with inelastic demand. For such seller, it remains optimal to fully reveal the realization of supply before competitive bids are submitted as long as the price paid by non-competitive bidders is a weakly increasing function of the market clearing price.

**Corollary 6. [Optimality of Information Disclosure with Noncompetitive Demand]** Suppose that noncompetitive demand is \( Q_{nc} \sim F_{nc} \), and that competitive supply is \( \overline{Q} - Q_{nc} \). If the seller allocates quantity \( Q_{nc} \) to noncompetitive bidders at price \( p_{nc}(p^*) \), which is weakly increasing in the market-clearing price \( p^* \), the seller’s revenue is maximized when the seller commits to fully-reveal the realization of noncompetitive demand.

The assumption that the per-unit price paid by noncompetitive bidders is increasing in the market-clearing price allows for noncompetitive demand to be filled at a fixed price, or at the market-clearing price, or at a constant markup over the market-clearing price (among
other possibilities).\footnote{In spot electricity markets in which the non-competitive electricity consumers pay exogenous prices which depend neither on the bids submitted nor the market clearing price in electricity auctions for suppliers, our Theorem 6 implies that the auctioneer wants to reveal the consumers’ demand to the suppliers bidding in the auction.} In light of Theorem 6, Corollary 6 is straightforward to prove. The seller’s revenue from competitive bidders is highest when supply is announced before bids are submitted. Moreover, announcing available supply weakly increases the market-clearing price, since bids are below marginal values except at the maximum feasible quantity (1). Then announcing the realization of supply increases the expected revenue from competitive bidders, and also increases the \textit{ex post} revenue from noncompetitive bidders.

On the other hand, the incentives of noncompetitive bidders, whose bids generate non-competitive demand, are opposed to those of the auctioneer. These noncompetitive bidders would (if possible) commit to not reveal their bids prior to the submission of the competitive bidders’ bids because the revelation of noncompetitive demand weakly increases the market-clearing price \textit{ex post}, in turn increasing the per-unit price paid by noncompetitive bidders.

\section*{4.3 Discussion of Transparency and Disclosure Results}

The optimality of transparency and full revelation hinges on using the pay-as-bid format. In uniform-price auctions (cf. Section 5.1 for their definition), it can be optimal to randomize supply and not disclose the realization of randomness; the proof of Lemma 1 implies that a wide range of randomizations might be optimal. One reason to use randomization is to prevent a form of tacit collusion that has been observed in uniform price auctions. For instance, Harbord and Pagnozzi [2014] discuss the revelation of demand information in uniform-price procurement auctions for Colombia and New England power generation capacity and Schwenen [2015] discusses uniform-price procurement for New York power capacity; both of these papers show that the price in these auctions is set by one large bidder, de facto acting as a monopolist, while the other bidders submit bids that are too low to be profitably undercut (these papers study procurement auctions, in which bidding low corresponds to bidding high in our model). All bidders gain from the ability of the large bidder to set the market-clearing price as a monopolist, and increasing the randomness of supply could benefit the seller by breaking this equilibrium. Analogous equilibria do not occur in pay-as-bid, because the fringe bidders would need to pay their high bids (or sell at the low bids).

The transparency result substantially simplifies the seller’s optimization problem. The
problem becomes one of setting reserve price \( R \) and deterministic supply \( Q \) so as to maximize

\[
\mathbb{E}_s[\pi] = \Pr\left(v\left(\frac{Q}{n}; s\right) \geq R\right) \mathbb{E}
\left[v\left(\frac{Q}{n}; s\right) \mid v\left(\frac{Q}{n}; s\right) \geq R\right] Q
+ \Pr\left(v\left(\frac{Q}{n}; s\right) < R\right) R \mathbb{E}\left[n v^{-1}(R; s) \mid v\left(\frac{Q}{n}; s\right) < R\right].
\]

When signal \( s \) comes from an atomless distribution on a subset of \( \mathbb{R} \) and the bidders’ marginal values are increasing in the signal, the seller can separately maximize the reserve \( R^* \) conditional on low signals \( s < \hat{s} \) and supply \( Q^* \) conditional on high signals \( s \geq \hat{s} \), where \( \hat{s} = \inf\{s : v(Q^*/n; \hat{s}) \geq R^*\} \). The separability allows us to solve for optimal auctions, as illustrated in the following.

**Example 1.** Suppose that the common signal \( s \) is distributed uniformly on \((\underline{s}, \bar{s})\) and \( v(q; s) = s - \rho q \) for some constants \( \rho, \underline{s}, \bar{s} > 0 \) such that \( \bar{s} > \underline{s} \geq \rho \bar{Q}/n \). Then bidding strategies are linear, the optimal reserve price is \( R^* = \frac{\bar{s}+3\underline{s}}{8\rho} \), optimal supply is \( Q^* = \left(\frac{3\bar{s}+\underline{s}}{8\rho}\right)n \), and the resulting expected revenue is \( \frac{n}{2\rho} \left(\frac{m^2}{2} + \frac{3V}{8}\right) \) where \( m = \frac{\bar{s}+\underline{s}}{2} \) is the mean and \( V = \frac{(\bar{s}-\underline{s})^2}{12} \) the variance of the signal distribution.

In this example, the seller’s revenue is increased by mean-preserving spreads of the distribution of bidders’ values. The monotonicity of revenue in mean-preserving spreads obtains beyond this example, and does not hinge on the value distribution being uniform. We discuss such comparative statics in our supplementary note [Pycia and Woodward, 2020].

## 5 The Auction Design Game: Pay-as-Bid Dominates Uniform-Price

Sellers of homogeneous goods are not constrained to use pay-as-bid auctions. In practice, sellers usually choose between implementing a pay-as-bid auction or implementing a uniform-price auction, and which of these two formats is preferred remains an important open question.\(^{57}\) The literature which compares these formats—see our introduction for a discussion—does so without taking the seller’s endogenous choices into account. In this section we introduce the *auction design game*, in which we explicitly model the seller’s choice.

\(^{57}\)From a theoretical perspective, we might be also interested in the question what general selling mechanism is optimal. When bidders have the same information the seller can ask all bidders to report their private information and set each bidder’s allocation and payment in a way that fully extract the surplus of that among announced types that maximizes the seller’s revenue. Furthermore, when bidders’ private information is merely correlated, the seller can extract nearly the full surplus using Crémer-McLean mechanisms, cf. Myerson [1981] and Crémer and McLean [1988].
between pay-as-bid and uniform-price formats, as well as among supply distributions and reserve prices, as an extensive-form game.

This auction design game has two stages. In the first stage, the seller commits to a reserve price, a distribution of supply, and the auction format (pay-as-bid or uniform-price). We also consider constrained design games in which the auction format is fixed; we refer to these as pay-as-bid design game and uniform-price design game. In the second stage, bidders participate in the specified auction.\footnote{We consider perfect Bayesian equilibria of these games. This structure allows us to compare outcomes of optimally designed pay-as-bid and uniform-price auctions, and to discuss the economic implications of mechanism selection. Our main insight is that choosing pay-as-bid is weakly dominant for the seller.}

Our main insight is that choosing pay-as-bid is weakly dominant for the seller.

5.1 Uniform-Price Auctions

As discussed above, uniform-price auctions are the main alternative to the pay-as-bid auction format. In the uniform-price auction, the space of feasible bids, the market-clearing price \( p^\star \), and allocations \( q_i \) are defined in the same way as in pay-as-bid (see Section 2). The only feature distinguishing the two formats is the bidders’ payment rule: instead of paying their own bids, in the uniform-price format each bidder \( i \) pays a constant market price per unit, hence bidder \( i \)’s payment is \( p^\star q_i \).

As mentioned in Section 4.3, in a uniform-price auction it may be optimal to commit to random supply. A key reason this might happen is the failure of equilibrium uniqueness in uniform price. Because bidders’ continuation equilibrium can be selected based on the chosen distribution of supply, it is possible that choosing deterministic supply will yield lower revenue than random supply: when bidders play a low-revenue equilibrium when supply is deterministic (or close to deterministic), and play a high-revenue equilibrium otherwise, the seller may optimally concentrate the supply distribution around the deterministic optimum while retaining some randomness to ensure that bidders submit aggressive bids. The construction of such equilibria relies on the value space being rich in the following sense: the set \( \{ s : v(Q^\star/n; s) > R^\star \} \) has positive probability for all deterministic supply and reserve pairs \((Q^\star, R^\star)\) that maximize monopoly revenue (4).\footnote{The bid functions \( b^i(s; R, F) \) depend on the bidders’ signal as well as the auction format and the reserve prices \( R \) and supply distributions \( F \) chosen by the seller. When there is no risk of confusion, when referring to the bids on the equilibrium path we sometimes suppress the seller’s choices.}

\begin{lemma}

[Quantity and Reserve in Uniform Price] Suppose the value space is rich and let \( R^{PAB} \) and \( Q^{PAB} \) be optimal reserve and supply in the pay-as-bid design game. There
\end{lemma}
is $\varepsilon > 0$ such that for all reserve prices $R \in [R^{\text{UPAB}} - \varepsilon, R^{\text{UPAB}} + \varepsilon]$ and all supply distributions $F$ with support in $[Q^{\text{UPAB}} - \varepsilon, Q^{\text{UPAB}} + \varepsilon]$, there is an equilibrium of the uniform-price design game in which the designer selects reserve $R$ and supply distribution $F$.

The proof builds on the construction of two equilibria classes:

- Robust equilibrium, defined as a profile of strategies that is an equilibrium for all distributions of supply; the existence and uniqueness of such an equilibrium follows from Klemperer and Meyer [1989]; and

- Semi-truthful equilibria, defined as equilibria at which $b^{\text{UPA}}(Q^R/n; s) = v(Q^R/n; s)$.

Appendix E.1 constructs both these equilibria classes and shows that, under the richness assumption, the expected revenue from the robust equilibrium following any reserve and supply distribution is strictly lower than (and bounded away from) the expected revenue from a semi-truthful equilibrium following reserve $R^{\text{UPAB}}$ and deterministic supply $Q^{\text{UPAB}}$. The perfect Bayesian equilibrium implementing reserve $R$ and supply distribution $F$ is then constructed as follows. If the seller sets $R$ and $F$ then, in the continuation game, bidders play the constructed semi-truthful equilibrium. If the seller sets different reserve or different distribution of supply then, in the continuation game, the bidders play the robust equilibrium, which has comparatively low bids. As $\varepsilon$ goes to 0, the expected revenue in the semi-truthful continuation equilibrium approximates that in the semi-truthful continuation equilibrium following reserve $R^{\text{UPAB}}$ and supply $Q^{\text{UPAB}}$. As the difference between the expected revenue in robust and semi-truthful equilibria following $R^{\text{UPAB}}$ and $Q^{\text{UPAB}}$ is bounded away from zero, for all $R$ and $F$ within sufficiently small $\varepsilon$ of $R^{\text{UPAB}}$ and $Q^{\text{UPAB}}$ (respectively), the expected revenue from setting $R$ and $F$ is strictly higher than the revenue from any other reserve and supply distribution.

### 5.2 Revenue

For the pay-as-bid auction, Theorem 2 states that equilibrium bids are essentially unique conditional on the distribution of supply, and Theorem 5 states that optimal supply is deterministic. Together these theorems imply that equilibrium revenue is unique in the pay-as-bid design game.

**Corollary 7. [Revenue in Pay-as-Bid Design Game]** In the pay-as-bid design game with symmetrically informed bidders, the perfect Bayesian equilibrium revenue is uniquely determined and the seller can achieve it by setting optimal deterministic supply.
Revenue analysis of the uniform-price design game is more complicated: as we have seen in the previous subsection randomness might be optimal on the path of a particular equilibrium. Despite this we show in Lemma 14 in Appendix E that the maximum revenue in uniform-price design game is obtained in a perfect Bayesian equilibrium in which the seller sets the same reserve price and deterministic supply as in revenue-maximizing pay as bid. In consequence, any equilibrium of the uniform-price game generates weakly less revenue than the unique expected revenue in any equilibrium of the pay-as-bid design game.

**Theorem 7. [Revenue Comparison of Design Games]** The expected revenue of the pay-as-bid design game is weakly greater than the expected revenue in any equilibrium of the uniform-price design game.

The revenue comparison is strict for all uniform-price equilibria in which bidders are not semi-truthful. The non-semi-truthful equilibria are typical in the sense that in the uniform-price auction, for any reserve $R$, supply distribution $F$, and signal $s$, the set of prices at maximum supply $Q^R$ that are supportable in equilibrium is the interval $[R, v(Q^R(s)/n; s)]$. In particular, robust equilibria are not semi-truthful and the ranking of pay as bid and uniform price becomes strict for robust equilibria. At the same time, there is a semi-truthful equilibrium of the uniform-price design game that generates the same expected revenue as the unique equilibrium revenue of the pay-as-bid design game. Appendix E establishes these claims and proves the theorem.

Theorem 7 implies that in the auction design game in which the designer chooses either a pay-as-bid or uniform-price format, and its reserve price and supply distribution, the seller will either implement a pay-as-bid auction or, expecting the bidders to bid semi-truthfully in uniform price, is indifferent between the two formats.

**Corollary 8. [Revenue Equivalence Across Perfect Bayesian Equilibria]** All perfect Bayesian equilibria of the auction design game are revenue equivalent. Furthermore, the seller either implements a pay-as-bid auction or is indifferent between the pay-as-bid and uniform-price auctions.

### 5.3 Welfare

The comparison of outcomes other than revenue—e.g., bidders’ payoffs and expected surplus—depends on the perfect Bayesian equilibrium played.

**Theorem 8. [Ambiguous Bidder Welfare Comparison]** If the value space is rich then the uniform-price design game admits perfect Bayesian equilibria in which the payoff of all
bidder types is strictly higher and perfect Bayesian equilibria in which the payoff of all bidder types is strictly lower than in the unique equilibrium of the pay-as-bid design game.

The reason for this ambiguity is that the quantity sold and reserve price in optimal uniform price can be strictly higher, the same, or strictly lower than in pay as bid, depending on the equilibrium in uniform price, as we have seen in Lemma 1 above. If the reserve price $R^\text{UP}$ in the uniform-price design game is strictly lower than the optimal pay-as-bid reserve $R^\text{PAB}$ and the supply $Q^\text{UP}$ in uniform price is deterministic and strictly higher than the optimal pay-as-bid supply $Q^\text{PAB}$, then there is an equilibrium of uniform price in which all bidder types pay $R^\text{UP}$ for each unit they buy their payoffs are strictly higher than in pay as bid. If, conversely, $R^\text{UP} > R^\text{PAB}$ and $Q^\text{UP} < Q^\text{PAB}$ then, irrespective of the equilibrium bids in uniform price, all bidder types have lower payoffs in uniform price than in the pay-as-bid design game.\(^{60}\) In the latter case, for distributions of bidders’ value functions for which the solution to the monopoly problem (4) is unique (a generic property), the seller’s revenue is also strictly lower in uniform price. Generically, there are thus equilibria of the uniform-price design that are strictly worse for all market participants than the essentially unique equilibrium of the pay-as-bid design game, but not vice versa (cf. Theorem 7).

6 Relationship to Empirical Findings

As discussed in the Introduction, an extensive empirical literature studies the use of the pay-as-bid and uniform-price auctions in real-world settings. Our model and main results correspond to empirical features observed across these studies. First, while empirical work provides no clear guidance on which of the pay-as-bid or uniform-price auction formats raises greater expected revenue in general, Table 1 shows that, across studies where supply randomness is reported, pay-as-bid dominates when supply randomness is small. This observation aligns with our transparency result (Theorem 5), which shows that when supply is deterministic the pay-as-bid auction raises strictly greater revenue than all but the seller-optimal equilibrium of the uniform-price auction.\(^{61}\)

An important prediction of our model is that bids are approximately flat when outcomes

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\(^{60}\) Ausubel et al. [2014] established that efficiency and revenue in uniform price can be higher or lower than in pay as bid depending on utility specification, assuming that the reserve price is zero and supply is unoptimized. As we study optimized auctions, there is no contradiction between the ambiguity they report and our revenue dominance, nor are our welfare comparisons implicit in theirs. The welfare ambiguity we uncover is driven by equilibrium selection and obtains for all utility specification in every model with rich values. In contrast, they provide examples of ambiguity that hinge on comparing equilibria between different model specifications and that rely on ex-ante asymmetries between bidders.

\(^{61}\) In our companion work, Pycia and Woodward [2022b], we show that this theoretical property remains approximately valid when bidders have asymmetric information.
are relatively certain (Corollary 2); conversely, when outcomes are relatively uncertain bidders will hedge against low allocations by bidding more aggressively for low quantities. Given a bidder’s uncertainty, flatness is a property of best-responses and does not hinge on the bids being in equilibrium. We can hence use this prediction to test the validity of the assumption that bidders are approximately symmetrically informed. Bid flatness has been observed in empirical analysis of Canadian [Hortaçsu and Sareen, 2005], South Korean [Kang and Puller, 2008], Chinese (Barbosa et al., 2020, and Yoshimoto, 2021, private communication), and Polish (Marszalec, 2017, and Marszalec, 2021, private communication) pay-as-bid treasury auctions, indicating that bidders face little relevant asymmetric information or other uncertainty in these auctions.

Our results on the auction design game suggest that the auctioneer’s choice of auction format will carry information about which auction format yields greater revenue. Cross-country comparisons find that both pay-as-bid and uniform-price auctions are popular (see OECD [2021] and Brenner et al. [2009] for treasury securities, and Maurer and Barroso [2011] and Del Río [2017] for electricity auctions), and our results provide a theoretical explanation for the popularity of the pay-as-bid format. Our Theorem 3 and Proposition 1 imply that, in large competitive markets, pay as bid and robust bids in uniform price will raise similar revenue, while in smaller markets pay as bid is likely to be revenue dominant. Of course, our predictions are only a baseline, and the auctioneer may be interested in outcomes beyond revenue.

Finally, Corollary 8 provides an explanation of the empirical finding that revenues in pay-as-bid are close to the counterfactual revenues in uniform price, as discussed in the Introduction. The explanation is two-fold. First, by Corollary 8, a revenue-maximizing seller weakly prefers the uniform-price format only if this format is equivalent to pay as bid. The South Korean Treasury auctions studied by Kang and Puller [2008] and U.S. Treasury auctions studied by Hortaçsu, Kastl, and Zhang [2018] run the uniform-price format and hence Corollary 8 provides a potential explanation of the revenue equivalence found in these papers. Second, the optimal pay-as-bid and uniform-price auctions generate the same revenue only in the seller-optimal equilibrium of the uniform-price auction and this is precisely the equilibrium in which bids are equal to marginal values at realized quantities. The latter equality is imposed in counterfactual revenue estimation of uniform-price auctions

62 The reason for this common information includes the presence of forward markets and close substitutes, as discussed in the Introduction.
63 Hortaçsu et al. [2018] observe flat bids in United States Treasury auctions, which are uniform-price.
64 Large market revenue equivalence, proved in Appendix F, is consistent with earlier large market results, cf. Swinkels [2001]. Our bid representations go further by making explicit the dependence of bids on the number of bidders (cf. Corollary 4).
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<thead>
<tr>
<th>Paper</th>
<th>Data</th>
<th>Method</th>
<th>$\sigma/\mu$</th>
<th># Bidders</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marszalec [2017]</td>
<td>Poland</td>
<td>PAB $\rightarrow$ CF UP</td>
<td>0.00%</td>
<td>12.3</td>
<td>PAB &gt; UP</td>
</tr>
<tr>
<td>Barbosa et al. [2020]</td>
<td>China</td>
<td>Controlled exp.</td>
<td>0.00%</td>
<td>35.2</td>
<td>PAB $\approx$ UP</td>
</tr>
<tr>
<td>Février et al. [2002]</td>
<td>France</td>
<td>PAB $\rightarrow$ CF UP</td>
<td>1.27%</td>
<td>20.8</td>
<td>PAB &gt; UP</td>
</tr>
<tr>
<td>Armantier and Sbai [2006]</td>
<td>France</td>
<td>PAB $\rightarrow$ CF UP</td>
<td>3.78%</td>
<td>19.0</td>
<td>UP &gt; PAB</td>
</tr>
<tr>
<td>Hattori and Takahashi [2022]</td>
<td>Japan</td>
<td>Natural exp.</td>
<td>11.00%</td>
<td>no data</td>
<td>PAB &gt; UP</td>
</tr>
<tr>
<td>Umlauf [1993]</td>
<td>Mexico</td>
<td>Natural exp.</td>
<td>11.16%</td>
<td>24.7</td>
<td>UP &gt; PAB</td>
</tr>
<tr>
<td>Mariño and Marszalec [2020]</td>
<td>Philippines</td>
<td>Natural exp.</td>
<td>17.60%</td>
<td>20.3</td>
<td>PAB &gt; UP</td>
</tr>
</tbody>
</table>

Table 1: Revenue comparisons between auction formats, in comparison to the standard deviation of noncompetitive demand scaled by mean aggregate supply ($Q$); “CF” is “counterfactual.”

in Hortaçsu and McAdams [2010] and Marszalec [2017] which assume truthful reporting in the uniform-price auction; as these papers discuss, the imposed assumption results in an upper bound on uniform-price revenue. The counterfactual assumption of truthful reporting in the uniform-price auction not only might bias expected revenues upward, but is likely to do so when supply randomness is high, cf. Klemperer and Meyer [1989]; when supply randomness is low pay as bid is approximately revenue equivalent to truthful bidding in uniform price. Our results thus suggest that the empirical ambiguity of cross-mechanism revenue comparison might be tied to sellers’ endogenous selection of auction format and to equilibrium selection in the empirical literature.

7 Conclusion

We have studied multi-unit auctions in an environment in which bidders have symmetric information, but the seller (or auction designer) is potentially much less informed; we consider the case of asymmetric information among bidders in our companion work Pycia and Woodward [2022b]. In this companion work we extend Theorem 1 to asymmetric information and we show that the equilibrium of this paper, without informational asymmetries among bidders, provides a lower bound on pay-as-bid revenues in the presence of informational asymmetries in the limit as the asymmetry of information vanishes. We hope that the tractability of our representation will stimulate future work on this important auction format. Wittwer [2017] discusses the intuition behind our representation.

65

66
are inherently ambiguous. In particular, it is possible that revenue-maximizing pay-as-bid auctions are not only revenue—but also welfare—superior to uniform-price auctions.

As part of our analysis we established revenue equivalence between revenue-maximizing pay-as-bid auctions and the revenue-maximizing equilibrium of uniform-price auctions. Our revenue equivalence benchmark—which we prove both for optimally-designed auctions and for deterministic supply—provides an explanation for the empirical findings of approximate revenue equivalence between the two formats by imposing that the revenue maximizing equilibrium obtains in uniform-price auctions; this is precisely the assumption that we show leads to theoretical revenue equivalence.

Our revenue comparison and equivalence results are consistent with the second-order details of empirical findings regarding multi-unit auction revenue; Table 1 relates revenue comparisons from the literature to normalized randomness in aggregate supply, and as expected given our results its shows that for small randomness pay-as-bid and uniform price are equivalent or pay-as-bid is revenue dominant, while for larger randomness either format can be revenue dominant.

References


67In the companion work Pycia and Woodward [2022b], we show that the results are robust to the presence of small asymmetric information among bidders.

68Table 1 summarizes all empirical studies for which we have the data allowing us to calculate the relative randomness of a single run of an auction. The randomness measures $\sigma/\mu$ are taken from either published work (Umlauf [1993], Février et al. [2002], and Armantier and Sbaï [2006]) or personal correspondence (Marszalec [2017], Barbosa et al. [2020], and Mariño and Marszalec [2020]).


Fei Li, Yangbo Song, and Mofei Zhao. Global manipulation by local obfuscation. 2020.


Elastic Supply

In the main text we (mostly) focus on pure strategy-equilibria and on designing a potentially stochastic supply distribution allowing for a separately set reserve price. Our essential insights remain valid if we allow mixed-strategy equilibria and potentially stochastic elastic supply curves.\footnote{As discussed in footnote 77, in the special case of our model with perfectly correlated types, \((s, \theta_i) = (s, 0)\), the seller can fully extract the bidders’ rents. Furthermore, in our model of asymmetrically informed bidders with demand curves coming from \(\delta\) support around commonly known (among bidders) demand, the seller could come within order-\(\delta\) of full extraction: even if idiosyncratic information is i.i.d., the seller can extract from the bidders the common part of the demand curve.}

We study the seller who selects a distribution over reserve prices, possibly correlated with the distribution of quantity. Let \(K(Q; R)\) be a supply-reserve distribution, giving the probability that realized quantity is \(\tilde{Q} \leq Q\) or the realized reserve price is \(\tilde{R} > R\),\footnote{In general, \(K(Q; R) = 1 - \Pr(\tilde{Q} > Q, \tilde{R} < R)\), and \(K\) is not a cumulative distribution function. In the absence of mass points, however, \(\Pr(\tilde{Q} \leq Q, \tilde{R} \leq R) = K(Q; R) - K(0; R)\), and the cumulative distribution function is in one-to-one correspondence with \(K\).}

\[
K(Q; R) = \Pr(\tilde{Q} \leq Q) + \Pr(\tilde{Q} > Q, \tilde{R} > R).
\]

Note that conditional on aggregate demand \(p(\cdot)\), \(K(Q; p(Q))\) is the probability that realized aggregate supply is below \(Q\): either realized supply is \(\tilde{Q} \leq Q\), or realized reserve is \(\tilde{R} > p(Q)\) and quantity is constrained. The following special cases illustrate the supply-reserve distribution \(K\):

- If \(K\) is equivalent to a random supply distribution \(F\) then \(K(Q, R) = F(Q)\);
- If \(K\) is equivalent to a random reserve distribution \(F^R\) then \(K(Q, R) = 1 - F^R(R)\);
- If \(K\) is equivalent to deterministic supply curve \(S\) then \(K(Q, R) = 1[S(R) < Q]\).

We allow an arbitrary distribution of the bidders’ common signal \(s\).

To key to extending our results to this environment is establishing the analogues of our uniqueness and optimality of deterministic supply results. Equilibrium uniqueness obtains
when the elastic supply curve is deterministic because an analogue of Theorem 1 obtains (see Appendix G for details of this and other proofs).

**Theorem 9. [Unique Pay-as-Bid Equilibrium]** If the elastic supply is deterministic then the pay-as-bid auction admits an essentially unique mixed-strategy equilibrium.

In this essentially unique mixed-strategy equilibrium, all bidders bid their marginal value on the last allocated unit for all units they receive; they can randomize over their bids on units the do not receive with no impact on equilibrium outcome.

Perhaps paradoxically, the main difficulty in proving the optimality of deterministic elastic supply lies in establishing this result for the case when the bidders’ common signal, $s$, is known to the seller—that is when it takes a constant value with probability 1.

**Lemma 2. [Deterministic Dominance when the Seller Knows Bidders’ Signal]** Suppose bidders’ information is known to the seller. Given any supply-reserve distribution $K$, there is a deterministic quantity $Q^*$ such that the pay-as-bid auction with fixed supply $Q^*$ raises greater revenue than the pay-as-bid auction with supply-reserve distribution $K$.

We prove this auxiliary complete-information result by studying an auxiliary problem in which a bidder’s bid satisfies a best-response first order condition but is not necessarily a best response to the random elastic supply and other bidders’ mixed strategies. We show that if—counterfactually—the seller was able to set the random supply-reserve distribution separately for this focal bidder, holding the other bidders’ behavior fixed, then the seller would optimize this part of the revenue by keeping the quantity allocated to the focal bidder constant and randomizing only over reserve prices. That is, analyzing constant supply and random reserve decouples the focal bidder’s best response from strategies of other bidders. Thus—given the symmetry of the problem—the seller is able to implement such a revenue maximizing scheme via a pay-as-bid auction with fixed supply and the same random supply distribution for all bidders. Leveraging the simplification brought by being able to restrict attention on random reserve only, we bound the maximum revenue of the seller by the revenue from a deterministic supply and reserve pay-as-bid (and uniform-price with identical supply and reserve).

Having shown that if the seller knew bidders’ common information, then she can do no better than set deterministic elastic supply so as to maximize the revenue, it remains to observe that the seller can obtain this revenue pointwise with an elastic supply curve. This observation relies on the following notion of regularity.

---

71 We also show a further technical property that—with arbitrarily small revenue loss—the reserve distribution can be so chosen that the focal bidder submits a strictly decreasing bid.
Definition 1. [Regular Demand] Let \( \mathcal{S} = \{(p^*, q^*) : \exists s, p^* \in \arg\max_p pv^{-1}(p; s), q^* = v^{-1}(p; s)\} \) be the set of optimal monopoly prices. Bidder values are regular if, for any \((p, q), (p', q') \in \mathcal{S}\), the inequality \( p' < p \) implies \( q' < q \).

Values are regular if the monopolist’s optimal price and quantity are in monotone correspondence.\(^{72}\) When values are increasing in signal \( s \), demand is regular when \( p + v^{-1}(p; s)/v_p^{-1}(p; s) \) is increasing in \( s \).\(^{73}\) Thus our regularity condition is similar to the regularity condition in [Myerson, 1981]. When bidder values are regular the seller can implement optimal reserve and quantity via an elastic supply function even though the seller does not know the bidders’ information.

Theorem 10. [Deterministic Auctions Are Optimal] When bidder values are regular then revenue in the pay-as-bid auction is maximized by implementing a deterministic supply curve. Any mixed-strategy equilibrium of the pay-as-bid auction with any random elastic supply raises weakly lower revenue than the unique equilibrium of pay-as-bid with optimal deterministic supply.

Because deterministic elastic supply is not only optimal in pay-as-bid, but also extracts the same revenue as if the seller knew bidders’ values, we can also conclude the following:

Theorem 11. [Pay-as-Bid Revenue Dominance] If bidder values are regular then the unique equilibrium of the optimal pay-as-bid auction raises weakly more revenue than any equilibrium in uniform-price auction with any supply-reserve distribution.

Furthermore, for a generic distribution of values there are multiple equilibria in uniform-price, and the revenue in a generic uniform-price equilibrium is strictly lower than the revenue in optimal pay-as-bid. This last point follows from the underpricing equilibrium constructions in, e.g., Back and Zender [1993] and LiCalzi and Pavan [2005].

Finally, our analysis of optimal elastic supply implies that an analogue of the information disclosure Theorem 6 remains true in under random elastic supply. Recall that in this theorem the quantity is exogenously realized and the seller has the ability to communicate this cap to the bidders. Because the optimal elastic supply is constructed point-by-point and hence does not depend on the quantity cap other than in the inelastic part of the supply when the cap is binding, in the current elastic supply setting the seller still wants to set the elastic supply (where possible) and fully reveal their private information.

\(^{72}\)Recall that we do not make any assumptions on the bidders’ type space, and in particular we do not require that demand increases with type.

\(^{73}\)To maximize profits, \( dpw^{-1}(p; s)/dp = 0 \), implying \( p + v^{-1}(p; s)/v_p^{-1}(p; s) = 0 \). If the left-hand side is increasing in \( s \), then \( p^* \) is increasing in \( s \). To have quantity also increasing in \( s \), we need \( dqv(q; s)/dq = 0 \), or \( qv_q(q; s) + v(q; s) = 0 \). Under monopoly, \( q = v^{-1}(p; s) \) and \( p = v(q; s) \), and the conditions for monotonicity in price and in quantity are equivalent.
Theorem 12. [Optimality of Information Disclosure with Elastic Supply] If the bidders’ values are regular then the seller’s expected revenue is maximized when the seller commits to fully reveal the realization of the elastic supply curve.
Supplementary Appendix (For Online Publication): Proofs

B Proof of Theorem 1 and Auxiliary Lemmas

In what follows, we denote the inverse hazard rate of aggregate supply by \( H = \frac{1-F}{f} \).

B.1 Proof of Theorem 1 (Minimum Market Price)

We allow mixed strategies and consolidate bidder-known uncertainty into \( \zeta_i = (s, \xi_i) \), where \( s \) is the signal observed by all bidders and \( \xi_i \) is a term indexing bidder \( i \)'s potentially-mixed strategy; thus bidder \( i \)'s bid is parameterized by \( \zeta_i \), hence \( b^i: [0, \overline{Q}] \times \text{Supp} \zeta_i \to \mathbb{R}_+ \). Where useful, we consider \( \zeta_i | s \) to hold fixed the common signal \( s \) while letting \( \xi_i \) vary.

The (essential) minimum market clearing price \( p \) and (essential) maximum receivable quantity \( q^i \), conditional on strategy profile \( (b^j)_{j=1}^n \), are defined as follows

\[
p(s) = \text{ess inf}_{Q, \zeta_i | s} p \left( Q; \left( b^j (\cdot; \zeta_j) \right)_{j=1}^n \right);
\]

\[
q^i (\zeta_i) = \text{ess sup}_{Q, \zeta_{-i} | s} q^i \left( Q; b^i (\cdot; \zeta_i), b^{-i} (\cdot, \zeta_{-i}) \right).
\]

Thus, when the bidding strategy profile is \( (b^j)_{j=1}^n \), the market clearing price is almost never below \( p(s) \) when the common signal is \( s \), and bidder \( i \)'s allocation is almost never above \( q^i (\zeta_i) \) when her type is \( \zeta_i \).

Lemma 3. In any equilibrium, conditional on common signal \( s \), at least \( n−1 \) bidders, with probability 1, bid their true value for their maximum receivable quantity. That is,

\[
\# \left\{ i : \Pr \left( b^i (\overline{q}^i (\zeta); \zeta) = v \left( \overline{q}^i (\zeta); s \right) \mid s \right) = 1 \mid s \} \geq n-1.
\]

Proof. For a given agent \( i \), common signal \( s \), and \( \lambda > 0 \), consider an alternative bidding strategy \( b^\lambda \) defined by

\[
b^\lambda (q; \zeta_i) = \begin{cases} 
    b^i (q; \zeta_i) & \text{if } b^i (q; \zeta_i) \geq b^i (\overline{q}^i (\zeta_i); \zeta_i) + \lambda, \\
    \min \{ b^i (\overline{q}^i (\zeta_i); \zeta_i) + \lambda, v (q; s) \} & \text{otherwise}.
\end{cases}
\]

The essential infimum is the highest value a random variable exceeds with probability one, \( \text{ess inf}_X f(X) = \sup \{ x : \Pr(X \geq x) = 1 \} \). The essential supremum is defined similarly.
Since \( b^i(\cdot; \zeta_i) \) is left-continuous, for small \( \lambda \) this deviation will award the agent all excess quantity above \( \sum_{j \neq i} q^j(b^j(\bar{q}(\zeta_i); \zeta_i) + \lambda; \zeta_j) \). Let \( q^*(\lambda; \zeta) \) be the quantity obtained under this deviation when, under the original strategy, \( q^i(\zeta) \) units would be obtained. Explicitly,

\[
q^* (\lambda; \zeta) = Q - \sum_{j \neq i} \varphi^j(b^j(\bar{q}(\zeta_i); \zeta_i) + \lambda; \zeta_j) = Q - \sum_{j \neq i} q^{ji} (\lambda; \zeta) ,
\]

where \( q^{ji}(\lambda; \zeta) = \varphi^j(b^j(\bar{q}(\zeta_i); \zeta_i) + \lambda; \zeta_j) \) is the quantity bidder \( j \) receives when the aggregate type profile is \( \zeta \) and bidder \( i \) implements bid \( b^\lambda \); note that \( q^{ji}(\lambda; \zeta) \) is the maximum quantity for which bidder \( i \) bids above \( b^i(\bar{q}(\zeta_i); \zeta_i) + \lambda \), which does not depend on \( \zeta_{-i} \), and denote this quantity by \( q^{i*}_i(\zeta_i) \). We will use the quantity \( q^*(\lambda; \zeta) \) to analyze the additional quantity the deviation yields above baseline,

\[
\Delta^L_i (\lambda; \zeta) = q^i (\zeta) - q^{ji} (\lambda; \zeta) , \quad \Delta^R_i (\lambda; \zeta) = q^* (\lambda; \zeta) - q^i (\zeta) , \quad \Delta^i (\lambda; \zeta) = \Delta^L_i (\lambda; \zeta) + \Delta^R_i (\lambda; \zeta) .
\]

Incentive compatibility requires that this deviation cannot be profitable, hence the additional costs must outweigh the additional benefits,

\[
\mathbb{E}_{Q, \zeta_i} \left[ \int_{q^i(\zeta_i)}^{q^*(\lambda; \zeta)} b^\lambda (x; \zeta_i) - b^i (x; \zeta_i) \, dx \quad \left| \quad q_i \geq q^{i*}_i (\zeta_i) \right. \right] \\
\geq \mathbb{E}_{Q, \zeta_i} \left[ \int_{q^* (\lambda; \zeta)}^{q^i (\zeta)} v(x; \zeta_i) - b^\lambda (x; \zeta_i) \, dx \quad \left| \quad q_i \geq q^{i*}_i (\zeta_i) \right. \right] .
\]

Importantly, this inequality must hold both ex ante and interim, unconditional on \( \theta_i \). Because bids are weakly decreasing, the left-hand expectation is bounded above by

\[
\mathbb{E}_{Q, \zeta_i} \left[ \int_{q^i(\zeta_i)}^{q^*(\lambda; \zeta)} b^\lambda (x; \zeta_i) - b^i (x; \zeta_i) \, dx \quad \left| \quad q_i \geq q^{i*}_i (\zeta_i) \right. \right] \\
\leq \mathbb{E}_{Q, \zeta_i} \left[ \int_{q^i(\zeta_i)}^{q^i(\zeta)} b^i (\bar{q}^i (\zeta_i); \zeta_i) + \lambda - b^i (\bar{q}^i (\zeta_i); \zeta_i) \, dx \quad \left| \quad q_i \geq q^{i*}_i (\zeta_i) \right. \right] \\
= \lambda \mathbb{E}_{Q, \zeta_{-i}} \left[ \Delta^L_i (\lambda; \zeta) \right] q_i \geq q^{i*}_i (\zeta_i) \right] .
\]

As marginal values are Lipschitz in quantity and \( b^i(\bar{q}^i (\zeta_i); \zeta_i) < v^i(\bar{q}^i (\zeta_i); s) \) by assumption,
the right-hand expectation is bounded above by \((M\) is the Lipschitz modulus of \(v\))

\[
\mathbb{E}_{Q,\zeta}|_{s}\left[\int_{q^{i}(\zeta)} v(x;\zeta) - b^{\lambda}(x;\zeta) \, dx \right] q_{i} \geq q^{i}_{\lambda}(\zeta) \right]
\geq \mathbb{E}_{Q,\zeta}|_{s}\left[\int_{q^{i}(\zeta)} v^{i}(\bar{q}^{i}(\zeta);s) - \left(x - q^{i}(\zeta)\right) M - \left(b^{i}(\bar{q}^{i}(\zeta);\zeta) + \lambda\right) \right) \, dx \right] q_{i} \geq q^{i}_{\lambda}(\zeta) \right]\]

\[
\geq \mathbb{E}_{Q,\zeta}|_{s}\left[\frac{1}{2} \left(\mu(\zeta) - \lambda\right) \Delta_{R}^{i}(\lambda;\zeta) q_{i} \geq q^{i}_{\lambda}(\zeta) \right],
\]

where \(\mu(\zeta) = v^{i}(\bar{q}^{i}(\zeta);s) - b^{i}(\bar{q}^{i}(\zeta);\zeta)\). If it is the case that \((\mu(\zeta) - \lambda)/M \leq \Delta_{R}^{i}(\lambda;\zeta)\) for all \(\lambda\), then it is impossible that the overall inequality is satisfied for all \(\lambda\) (its left-hand side converges to zero in \(\lambda\), while the right-hand side converges to a strictly positive value) and incentive compatibility is violated. Therefore we assume that the \(\min\{\cdot, \cdot\}\) resolves to \(\Delta_{R}^{i}(\lambda;\zeta)\). Then the overall inequality implies

\[
\lambda \mathbb{E}_{Q,\zeta}|_{s}\left[\Delta_{L}^{i}(\lambda;\zeta) q_{i} \geq q^{i}_{\lambda}(\zeta) \right] \geq \mathbb{E}_{Q,\zeta}|_{s}\left[\frac{1}{2} \left(\mu(\zeta) - \lambda\right) \Delta_{R}^{i}(\lambda;\zeta) q_{i} \geq q^{i}_{\lambda}(\zeta) \right].
\]

Since \(\Delta_{R}^{i}(\lambda;\zeta)\) is bounded, there is \(m^{i}(\lambda)\) such that

\[
\lambda \mathbb{E}_{Q,\zeta}|_{s}\left[\Delta_{L}^{i}(\lambda;\zeta) q_{i} \geq q^{i}_{\lambda}(\zeta) \right] \geq \frac{1}{2} \left(m^{i}(\lambda) - \lambda\right) \mathbb{E}_{Q,\zeta}|_{s}\left[\Delta_{R}^{i}(\lambda;\zeta) q_{i} \geq q^{i}_{\lambda}(\zeta) \right].
\]

For any \(i\), any \(\lambda\), and any \(\kappa > 0\), there is \(\Lambda^{i}(\lambda,\kappa) > 0\) such that

\[
\Lambda^{i}(\lambda,\kappa) \leq \frac{1}{2} \left(m^{i}(\lambda) - \lambda\right) \kappa.
\]

The term \(m^{i}(\lambda)\) can be specified so that \(m^{i}(\lambda) - \lambda\) is decreasing in \(\lambda\), so if \(\Lambda^{i}(\lambda,\kappa) < (m^{i}(\lambda) - \lambda)\kappa/2\), then \(\Lambda^{i}(\lambda,\kappa) < (m^{i}(\lambda') - \lambda')\kappa/2\) for all \(\lambda' > \lambda\). Then let \(\bar{\Lambda} = \min\{\Lambda^{i}(\lambda,\kappa) : \Pr_{\zeta}(b^{i}(\bar{q}^{i}(\zeta);\zeta) > v^{i}(\bar{q}^{i}(\zeta);s)\mid s) > 0\}\). For any such \(\kappa, \bar{\Lambda}\), it must be that

\[
\kappa \mathbb{E}_{Q,\zeta}|_{s}\left[\Delta_{L}^{i}(\bar{\Lambda};\zeta) q_{i} \geq q^{i}_{\lambda}(\zeta) \right] \geq \mathbb{E}_{Q,\zeta}|_{s}\left[\Delta_{R}^{i}(\bar{\Lambda};\zeta) q_{i} \geq q^{i}_{\lambda}(\zeta) \right].
\]

Define bidder \(j\) with type \(\zeta_{j}\) to be relevant given price \(p\) (and common signal \(s\)) if \(b^{i}(\bar{q}^{i}(\zeta_{j});\zeta_{j}) \leq p < v^{i}(\bar{q}^{i}(\zeta_{j});s)\). Fixing price \(p\) and summing the above incentive inequality
over all relevant agents gives

$$
\kappa \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s} \left[ \Delta^j_L (\bar{\alpha}; \zeta) \right| q_j \geq q^j_\alpha (\zeta_j)] \\
\geq \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s} \left[ \Delta^j_R (\bar{\alpha}; \zeta) \right| q_j \geq q^j_\alpha (\zeta_j)] \\
= \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s} \left[ \Delta^j (\bar{\alpha}; \zeta) \right| q_j \geq q^j_\alpha (\zeta_j)] - \mathbb{E}_{Q,\zeta|s} \left[ \Delta^j_L (\bar{\alpha}; \zeta) \right| q_j \geq q^j_\alpha (\zeta_j)] .
$$

Thus,

$$
(k + 1) \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s} \left[ \Delta^j_L (\bar{\alpha}; \zeta) \right| q_j \geq q^j_\alpha (\zeta_j)] \geq \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s} \left[ \Delta^j (\bar{\alpha}; \zeta) \right| q_j \geq q^j_\alpha (\zeta_j)] .
$$

By definition, \( \Delta^j(\bar{\alpha}; \zeta) = Q - q^j_\alpha (\zeta) - \sum_{k \neq j} q^{kj}(\bar{\alpha}; \zeta) \equiv Q - \bar{Q}^j(\bar{\alpha}; \zeta) \) and \( \Delta^j_L(\bar{\alpha}; \zeta) = q^j(\zeta) - q^j_\alpha (\bar{\alpha}; \zeta) \). Furthermore,

$$
\sum_{j \text{ relevant}} q^j(\zeta) - q^j_\alpha (\zeta_j) \leq \sum_{j \text{ relevant}} q^j(\zeta) - q^j_\alpha (\zeta_j) = Q - Q(p + \delta) .
$$

Then it follows that

$$
\kappa + 1 \geq \# \{ j \text{ relevant} \} .
$$

Since \( \kappa > 0 \) may be arbitrarily small, it follows that there is at most one relevant bidder; i.e., there is at most a single bidder \( i \) such that \( \Pr(b^i(\bar{q}^i(\zeta_i); \zeta_i) < v(\bar{q}^i(\zeta_i); s)) < 1. \)

**Lemma 4.** For all bidders \( i \) and all bidder-common signals \( s \),

$$
\Pr \left( b^i(\bar{q}^i(\zeta_i); \zeta_i) = v(\bar{q}^i(\zeta_i); s) \right| s) = 1.
$$

**Proof.** Fix a common signal \( s \). Lemma 3 shows that at least \( n - 1 \) bidders \( j \) are such that \( b^j(\bar{q}^j(\zeta_j); \zeta) = v(\bar{q}^j(\zeta_j); s) \) with probability 1. If all \( n \) bidders’ bids satisfy this condition, the desired result follows immediately from market clearing. Otherwise, there is some bidder \( i \) such that \( b^i(\bar{q}^i(\zeta_i); \zeta) < v(\bar{q}^i(\zeta_i); s) \) with \( \zeta_i \) strictly positive probability. We show that (i) this bidder’s bid must be constant in a neighborhood of \( \bar{q}^i(\zeta_i) \), (ii) with \( \zeta_{-i} \) strictly positive probability, opposing bidders’ bids are asymptotically flat near \( \bar{q}^i(\zeta_i) \), and (iii) this implies that bidder \( i \) has a strict incentive to increase her (flat) bid near \( \bar{q}^i(\zeta_i) \).

Let bidder \( i \) and parameter \( \zeta_i \) be such that \( b^i(\bar{q}^i(\zeta_i); \zeta_i) = p < v(\bar{q}^i(\zeta_i); s) \), and assume that \( b^i \) is strictly decreasing in a neighborhood to the left of \( \bar{q}^i(\zeta_i) \). For \( \lambda > 0 \), define an
alternate bid \( b^\lambda \),

\[
b^\lambda (q) = \begin{cases} 
b^i (q; \zeta_i) & \text{if } b^i (q; \zeta_i) \geq p + \lambda, \\
p + \lambda & \text{otherwise.}
\end{cases}
\]

Since \( b^i(\overline{\theta}^i(\zeta_i); \zeta_i) < v(\overline{\theta}^i(\zeta_i); s) \) and we analyze small \( \lambda > 0 \), we may assume that \( \lambda \) is small enough that for any feasible quantity \( q \), \( b^\lambda(q) \leq v(q; s) \). Then whenever the market clearing price would be \( p < p + \lambda \) if bidder \( i \) submitted bid \( b^i \), the market clearing price will be \( p + \lambda \) if she submits bid \( b^\lambda \) instead. Further, bidder \( i \) receives the full residual supply,

\[
q_i^\lambda = Q - \sum_{j \neq i} \overline{\varphi}^j(p + \lambda; \zeta_j).
\]

The utility gain associated with bid \( b^\lambda \) versus bid \( b^i \) is bounded below by

\[
\mathbb{E}_{Q,\zeta,\lambda} \left[ \int_q^{Q - \sum_{j \neq i} \overline{\varphi}^j(p + \lambda; \zeta_j)} v(x; s) - (p + \lambda) \, dx \right] \quad q \geq \overline{\varphi}^i(p + \lambda; \zeta_i) \quad (5)
\]

Because bidder \( i \)'s opponents all have \( \Pr(b^i(\overline{\theta}^i(\zeta_j); \zeta_j) = v(\overline{\theta}^i(\zeta_j); s)) = 1 \), and bids are below values and values are Lipschitz continuous, there is \( M > 0 \) such that \( \overline{\varphi}^i(\zeta_j) - \overline{\varphi}^i(p + \lambda; \zeta_j) > M\lambda \) with probability 1 for all \( j \neq i \). Then, letting \( \lambda < v(\overline{\theta}^i(\zeta_i); s) - b^i(\overline{\theta}^i(\zeta_i); \zeta_i) \), the bound in 5 is in turn bounded below by

\[
\mathbb{E}_{Q,\zeta,\lambda} \left[ \int_q^{Q - \sum_{j \neq i} \overline{\varphi}^j(\zeta_j)} (Q - \overline{\varphi}^i(\zeta_j)) + (n - 1) M\lambda \right] v(x; s) - (p + \lambda) \, dx - (q - \overline{\varphi}^i(p + \lambda; \zeta_i)) \lambda q \geq \overline{\varphi}^i(p + \lambda; \zeta_i) \lambda > 0.
\]

In the above we rely on the fact that the minimum market clearing price is obtained when aggregate supply is maximized. Since \( b^\lambda \) yields higher expected utility than \( b^i \) when \( \lambda > 0 \) is small, \( b^i \) is not a best response, and therefore any best response \( b^i \) must be constant in a neighborhood of \( \overline{\theta}^i(\zeta_i) \), if \( b^i(\overline{\theta}^i(\zeta_i); \zeta_i) < v(\overline{\theta}^i(\zeta_i); s) \).

Define \( \tilde{q}^i(\zeta_i) = \frac{p}{\varphi^i(p; \zeta_i)} \) to be the left endpoint of the flat interval of bidder \( i \)'s bid, containing \( \overline{\theta}^i(\zeta_i) \). Without loss of generality, we may assume that \( b^i(q; \zeta_i) = p \) for all \( q > \tilde{q}^i(\zeta_i) \) whenever \( b^i(\overline{\theta}^i(\zeta_i); \zeta_i) < v(\overline{\theta}^i(\zeta_i); s) \): extending the flat portion of the bid function...
either does not affect allocation, or (by market clearing) increases allocation to some \( q \) such that \( v(q; s) > p \). Since \( \Pr(b_i'(\bar{q}^i(\zeta_i); \zeta_i) < v(\bar{q}^i(\zeta_i); s)|s > 0 \) and \( \bar{q}^i(\zeta_i) < \bar{q}^i(\zeta_i) \) for all \( \zeta_i \) with \( b_i'(\bar{q}^i(\zeta_i); \zeta_i) < v(\bar{q}^i(\zeta_i); s) \), it follows that \( \Pr(p(Q, \zeta) = p|s > 0 \). Consider a bidder \( j \neq i \) and type \( \zeta_j \) such that \( b_i'(\bar{q}^i(\zeta_j); \zeta_j) = v(\bar{q}^i(\zeta_j); s) = p; \) since \( \Pr(p(Q, \zeta) = p|s > 0 \), it must be that \( \Pr(q_j = \bar{q}^i(\zeta_j)|s > 0 \). If the bid \( b_i(\cdot; \zeta_j) \) is optimal, it must not be utility-improving to decrease the bid to \( b^{\lambda \mu} \), where\(^75\)

\[
b^{\lambda \mu}(q) = \begin{cases} 
b_i(q; \zeta_j) & \text{if } q < \bar{q}^i(\zeta_j) - \lambda, \\
p + \mu & \text{otherwise.}
\end{cases}
\]

The bid \( b^{\lambda \mu} \) saves payment \( \int_{\bar{q}^i(\zeta_j) - \lambda}^{\bar{q}^i(\zeta_j)} b_i(q; \zeta_j) - (p + \mu)dq \) whenever \( q_j = \bar{q}^i(\zeta_j) \), but potentially reduces quantity when \( q_j \notin (\bar{q}^i(\zeta_j) - \lambda, \bar{q}^i(\zeta_j)) \). The change in utility from implementing bid \( b^{\lambda \mu} \) instead of bid \( b_i(\cdot; \zeta_j) \) is bounded below by

\[
\int_{\bar{q}^i(\zeta_j) - \lambda}^{\bar{q}^i(\zeta_j)} b_i(q; \zeta_j) - (p + \mu) dq \Pr(q_j = \bar{q}^i(\zeta_j)|\zeta_j) - \int_{\bar{q}^i(\zeta_j) - \lambda}^{\bar{q}^i(\zeta_j)} v(x; s) - b_i(x; \zeta_j) dx dG^i(q; b_i).
\]

The derivative of this expression with respect to \( \lambda \) must be weakly negative,

\[
\left(b_i(\bar{q}^i(\zeta_j) - \lambda; \zeta_j) - (p + \mu)\right) \Pr(q_j = \bar{q}^i(\zeta_j)|\zeta_j) \left(v(\bar{q}^i(\zeta_j) - \lambda; s) - b_i(\bar{q}^i(\zeta_j) - \lambda; \zeta_j)\right) \Pr(q_j \in (\bar{q}^i(\zeta_j) - \lambda, \bar{q}^i(\zeta_j))|\zeta_j) \leq 0.
\]

This inequality holds for all \( \mu > 0 \). Letting \( M \) be the Lipschitz modulus of \( v \), substituting in for \( b_i'(\bar{q}^i(\zeta_j); \zeta_j) = p \) means that the previous inequality implies

\[
\left(b_i(\bar{q}^i(\zeta_j) - \lambda; \zeta_j) - p\right) \Pr(q_j = \bar{q}^i(\zeta_j)|\zeta_j) - M\lambda \Pr(q_j \in (\bar{q}^i(\zeta_j) - \lambda, \bar{q}^i(\zeta_j))|\zeta_j) \leq 0
\]

\[
\iff -\frac{b_i(\bar{q}^i(\zeta_j); \zeta_j) - b_i(\bar{q}^i(\zeta_j) - \lambda; \zeta_j)}{\lambda} \leq \frac{M \Pr(q_j \in (\bar{q}^i(\zeta_j) - \lambda, \bar{q}^i(\zeta_j))|\zeta_j)}{\Pr(q_j = \bar{q}^i(\zeta_j)|\zeta_j)}.
\]

Taking the limit as \( \lambda \searrow 0 \), we obtain that \( b_i'(\bar{q}^i(\zeta_j); \zeta_j) = 0 \). Thus any bidder \( j \neq i \) with type \( \zeta_j \) such that \( b_i'(\bar{q}^i(\zeta_j); \zeta_j) = v(\bar{q}^i(\zeta_j); s) \) and \( \Pr(q_j = \bar{q}^i(\zeta_j)|\zeta_j) > 0 \) is such that \( b_i'(\bar{q}^i(\zeta_j); \zeta_j) = 0 \).

Now return to bidder \( i \) with type \( \zeta_i \) such that \( b_i'(\bar{q}^i(\zeta_i); \zeta_i) = v(\bar{q}^i(\zeta_i); s) \) and \( \bar{q}^i(\zeta_i) < \bar{q}^i(\zeta_i) < \bar{q}^i(\zeta_i) \), and consider the alternate bid function \( b^\lambda \) defined in the first portion of this proof. We now

\(^75\)The \( \mu \) term ensures that bidder \( j \) wins ties against the flat portion of bidder \( i \)'s bid; this term will be taken to zero and thus will have no marginal effect on utility.
place a slightly different bound on the utility gained by implementing bid $b^\lambda$ versus bid $b^i(\cdot; \zeta_i)$. Payments increase by at most $\overline{Q}\lambda$, with at most probability 1; and, whenever $q_i > \hat{q}(\zeta_i)$ under $b^i(\cdot; \zeta_i)$, bidder $i$ receives the full residual quantity $Q - \sum_{j \neq i} \overline{q}(p + \lambda; \zeta_j)$. Then a lower bound on the utility improvement generated by the alternate bid $b^\lambda$ (versus $b^i(\cdot; \zeta_i)$) is

$$E_{Q, \zeta_i} \left[ \int_q^{Q - \sum_{j \neq i} \overline{q}(p + \lambda; \zeta_j)} v(x; s) - px - \overline{Q}\lambda \right] q \geq \hat{q}(\zeta_i) \right].$$

For $b^\lambda$ to not be utility-improving, this expectation must be weakly negative. Dividing through by $\lambda$ and taking the limit at $\lambda \downarrow 0$ gives

$$E_{Q, \zeta_i} \left[ - \left( v \left( Q - \sum_{j \neq i} \overline{q}(\zeta_j); s \right) - p \right) \sum_{j \neq i} \overline{q}(p; \zeta_j) - \overline{Q}\lambda \right] q \geq \hat{q}(\zeta_i) \right] \leq 0.$$

By assumption, $v(Q - \sum_{j \neq i} \overline{q}(\zeta_j); s) > p$, and from the previous paragraph we have that $\overline{q}(p; \zeta_j) = -\infty$ with strictly positive probability. Then the above inequality cannot be satisfied. It follows that there is no bidder $i$ such that $\Pr(b^i(\overline{q}(\zeta_i); \zeta_i) < v(\overline{q}(\zeta_i); s) | \zeta_j) > 0$.

Lemma 4 states that for all bidders $i$, at the maximum quantity received with positive probability, the equilibrium bid must exactly equal the bidder’s true marginal value with probability 1. Consider a bidder playing the realization (of possibly mixed strategy) indicated by $\zeta_i$ such that $\overline{q}(\zeta_i) = \text{ess inf}_{\zeta_i} \overline{q}(\zeta_i)$; that is, bidding the mixed-strategy realization gives the lowest maximum allocation possible. By market clearing, $\sum_i \overline{q}(\zeta_i) = \overline{Q}R$, and by Lemma 4 and the fact that marginal values are strictly decreasing it follows that $\overline{q}(\zeta_i) = \overline{Q}R/n$. Then because bids are decreasing, it is only possible that $\overline{q}(\zeta_i) > \overline{Q}R/n$ if there is $q$ such that $b^i(q; \zeta_i) > v(q; \zeta_i)$, contradicting Lemma 6 below. It follows that for all bidders $i$ and almost all types $\zeta_i$, $\overline{q}(\zeta_i) = \overline{Q}R/n$, and Theorem 1 is demonstrated. In the case of pure strategies, the following corollary is immediate.

**Corollary 9.** In all pure-strategy equilibria of the pay-as-bid auction, the equilibrium minimum market-clearing price equals the marginal value for the per-capita maximum quantity,

$$p(\overline{Q}R; s) = \hat{v} \left( \overline{Q}R; s \right).$$
B.2 Pure strategy equilibrium derivation with symmetric bidder information

In this section we present the lemmas for our results on existence, uniqueness, and bid representation of pure strategy equilibria under symmetric bidder information. The arguments for deterministic supply was given in the main text, and here we focus on random supply. To simplify notation we will thus write \( v(q) \) in lieu of \( v(q; s) \) and \( b^i(q) \) in lieu of \( b^i(q; s) \).

Let us fix a pure-strategy candidate equilibrium \( (b^i)_{i=1}^n \). Recall that bid functions are weakly decreasing and (where useful) we may assume that they are right-continuous. Given equilibrium bids the market price (that is, the stop-out price) \( p(Q) \) is a function of realized supply \( Q \). In line with Appendix A, denote \( G^i(q; b^i) = \Pr(q^i \leq q | b^i) \); that is, \( G^i(q; b^i) \) is the probability that agent \( i \) receives at most quantity \( q \) when submitting bid \( b^i \) in the equilibrium considered. The monotonicity of bid functions implies that as long as \( b^i \) is an equilibrium bid, and given other equilibrium bids, the probability \( G^i(q; b^i) \) depends on \( b^i \) only through the value of \( b^i(q) \).

Our statements in the following results are generally about relevant quantities, such that \( G^i(q; b^i) < 1 \). For each bidder we ignore quantities larger than the maximum quantity this bidder can obtain in equilibrium; for instance, in the following lemmas, all bidders could submit identical flat bids above their values for units they never obtain. Accordingly, we restrict attention to relevant price levels \( p \), such that \( \Pr(p^* < p) > 0 \).

**Lemma 5.** For no relevant price level \( p \) are there two or more bidders who, in equilibrium, bid constant value \( p \) flat on some non-trivial intervals of quantities.

**Proof.** The proof resembles similar proofs in other auction contexts. Suppose agent \( i \) bids \( p \) on \( (q^i_t, q^i_s) \) and bidder \( j \) bids \( p \) on \( (q^j_t, q^j_s) \). Since the support of supply is \([0, \overline{Q}]\), it must be that \( G^i(q^i_t; b^i) > G^i(q^j_t; b^i) \) and \( G^j(q^j_t; b^i) > G^j(q^i_t; b^i) \). Let \( \overline{q}^i = \mathbb{E}_Q[q^i | p(Q) = b(q^i)] \); without loss of generality, we may assume that agent \( i \) is such that \( \overline{q}^i < q^i_s \). If \( v^i(\overline{q}^i) < b^i(q^i_s) \), the agent has a profitable downward deviation. The agent also has a profitable deviation if \( v^i(q^i_t) \geq b^i(q^i_s) \): she can increase her bid slightly by \( \lambda > 0 \) on \([q^i_t, q^i_s)\) (enforcing monotonicity constraints as necessary to the left of \( q^i_t \)), keeping her bid below value if necessary.\(^76\)

**Lemma 6.** Bids are below values: \( b^i(q) \leq v^i(q) \) for all relevant quantities, and \( b^i(q) < v^i(q) \) for \( q < \varphi^i(p(\overline{Q})) \).

**Proof.** Suppose that there exists \( q \) with \( b^i(q) > v^i(q) \); because \( b^i \) is monotonic and \( v^i \) is continuous, there must exist a range \((q_t, q_{t'})\) of relevant quantities such that \( b^i(q) > v^i(q) \)

\(^76\)Because we are conditioning on her expected quantity, we do not need to directly consider whether quantities are relevant.
for all \( q \in (q_\ell, q_r) \). The agent wins quantities from this range with positive probability, and hence the agent could profitably deviate to

\[ \hat{b}^i(q) = \min \{ b^i(q), v^i(q) \}. \]

Such a deviation never affects how she might be rationed, by the first part of this proof; hence it is necessarily utility-improving.

Now consider \( q < \varphi^i(p(Q)) \). If \( b^i(q) = v^i(q) \) then monotonicity of \( b^i \) and Lipschitz-continuity of \( v^i \) imply that for small \( \varepsilon > 0 \) winning units \([q - \varepsilon, q]\) brings per unit profit lower than \( M\varepsilon \), where \( M \) is the Lipschitz modulus of \( v \). By lowering the bid for quantities \( q' \in [q - \varepsilon, q + \varepsilon] \) to \( \hat{b}^i(q') = \min\{v^i(q) - \varepsilon, b^i(q')\} \), the utility loss from losing the relevant quantities is at most \( 2M\varepsilon^2 (G_i(q + \varepsilon; b^i) - G_i(q - \varepsilon; b^i)) \). Notice that the right-hand probability difference goes to zero as \( \varepsilon \) goes to zero. At the same time the cost savings from paying lower bids at quantities higher than \( q + \varepsilon \) is (at least) of order \( \varepsilon^2 \). Hence this deviation is profitable, and it cannot be that \( b^i(q) = v^i(q) \).

**Lemma 7.** The market clearing price \( p(Q) \) is strictly decreasing in supply \( Q \), for all \( Q \leq Q^R \).

**Proof.** We show first that the market clearing price is strictly decreasing in supply for all \( Q \) such that \( p(Q) > \inf_{Q'} p(Q') = \underline{p} \). We then show that \( p \) is strictly decreasing at \( Q^R \) as long as for any bidder \( i \) residual supply \( \sum_{j \neq i} \varphi^j(\cdot) \) has nonzero slope at \( \underline{p} \). Since Corollary 7 shows that \( b^i(\bar{q}^i) = \underline{p} \), Lemma 6 shows that bids are below values, and values are Lipschitz continuous, it follows that residual supply has nonzero slope at \( \underline{p} \), and therefore the market clearing price is strictly decreasing in \( Q \).

Since bids are weakly decreasing in quantity, the market price is weakly decreasing as a direct consequence of the market-clearing equation. If price is not weakly decreasing in quantity at some \( Q \), then a small increase in \( Q \) will not only increase the price, but will weakly decrease the quantity allocated to each agent. This implies that total demand is no greater than \( Q \), contradicting market clearing.

Lemma 5 is sufficient to imply that the market price must be strictly decreasing for all \( Q \) such that \( p(Q) > \underline{p} \) at every price level at which at least two bidders pay with positive probability for some quantity, at most one of the submitted bid functions is flat. Furthermore, for no price level \( p > \underline{p} \) that with positive probability a bidder pays for some quantity, we can have exactly one bidder, \( i \), submitting a flat bid at price \( p \) on an interval of relevant quantities.\(^{77}\) Indeed, in equilibrium bidder \( i \) cannot benefit by slightly reducing the bid on

\(^{77}\)We refer to any price level \( p \) that with positive probability a bidder pays for some quantity, as a relevant price level.
this entire interval; thus it must be that there is some other agent \( j \) whose bid function is right-continuous at price \( p \). If \( p = 0 \), all opponents \( j \neq i \) have a profitable deviation.\(^{78}\) If \( p > 0 \), we appeal to Lemma 6. Given that \( i \) submits a flat bid and the bids of bidder \( j \) are strictly below her values for some non-trivial subset of quantities at which her bid is near \( p \), bidder \( j \) can then profit by slightly raising her bid; this reasoning is similar to that given in the proof of Lemma 5.

We now show that \( p(\cdot) \) is strictly decreasing for all \( Q \). Otherwise, following Lemma 5, there is a bidder \( i \) who is submitting a flat bid at \( p \). Denote the left end of this bidder’s flat by \( q_i = \inf \{ q : b^i(q) = p \} \); by assumption, \( q_i < q_i \).\(^{79}\) Let \( \varepsilon, \lambda > 0 \) and define a deviation

\[
\hat{b}^{\varepsilon, \lambda}(q) = \begin{cases} 
    b^i(q) & \text{if } b^i(q) > p + \lambda, \\
    p + \lambda & \text{if } b^i(q) \leq p + \lambda \text{ and } q \leq q_i + \varepsilon, \\
    p & \text{otherwise.}
\end{cases}
\]

That is, \( \hat{b}^{\varepsilon, \lambda} \) is \( b^i \), with \( \lambda \) added for length \( \varepsilon \) at \( q_i \), and adjusting for the fact that bids must be monotone decreasing. Note that this deviation increases costs by at most \( (\varepsilon + (q_i - \varphi^i(p + \lambda)))\lambda \), with at most probability one. When \( q_i \in [q_i, q_i + \varepsilon] \), it increases the quantity allocation to (approximately) \( \max \{ q_i + \varepsilon, q + \lambda M \} \), where \( M \) is the slope of residual supply at the minimum price, \( M = \left| \sum_{j \neq i} \varphi^j(p) \right| \).\(^{80}\) Let \( \mu \equiv v^i(q_i + \varepsilon) - (p + \lambda) \); since bids are below values and values are strictly decreasing, \( \mu > 0 \) when \( \varepsilon \) and \( \lambda \) are sufficiently small. Then for the deviation to be nonoptimal, it must be that

\[
(\varepsilon + (q_i - \varphi^i(p + \lambda)))\lambda \geq \mathbb{E} \left[ \left( \max \left\{ \varepsilon, q + \frac{\lambda}{M} \right\} - q \right) \mu \bigg| q \in [q_i, q_i + \varepsilon] \right] 
= \mathbb{E} \left[ \left( \max \left\{ \varepsilon - q, \frac{\lambda}{M} \right\} \right) \mu \bigg| q \in [q_i, q_i + \varepsilon] \right].
\]

\(^{78}\) Here we work in a model in which marginal utilities on all possible units is strictly positive. We could dispense with the strict positivity assumption by allowing negative bids.

\(^{79}\) Because bidders are symmetric, it is not possible that \( q_i = 0 \): in this case, bidder \( i \) almost surely receives 0 utility ex post, which is not optimal.

\(^{80}\) Because we are ultimately letting \( \varepsilon \) and \( \lambda \) go to zero, this approximation is sufficient. Formally, we may consider \( M' < M \) and allow \( \delta \) to be small enough that the slope of residual supply never falls below \( M' \).
Letting $Q_{-i} = \sum_{j \neq i} q_j$, this can be rewritten as

\[
\left(\varepsilon + (q_i - \varphi^i (p + \lambda)) \right) \lambda \int_{q_i}^{q_i + \varepsilon} dF (q + Q_{-i}) \geq \int_{q_i}^{q_i + \varepsilon} \max \left\{ \varepsilon + q_i - q, \frac{\lambda}{M} \right\} \mu dF (q + Q_{-i}) \geq \int_{q_i}^{q_i + \varepsilon - \frac{\lambda}{M}} \frac{\mu \lambda}{M} dF (q + Q_{-i}) .
\]

The $\lambda > 0$ multipliers cancel; integrating through gives

\[
\left(\varepsilon + (q_i - \varphi^i (p + \lambda)) \right) \left( F (q_i + \varepsilon + Q_{-i}) - F (q_i + Q_{-i}) \right) \geq \frac{\mu}{M} \left( F \left( q_i + \varepsilon - \frac{\lambda}{M} + Q_{-i} \right) - F \left( q_i + Q_{-i} \right) \right)
\]

From here the argument is standard. For any $\varepsilon > 0$ there is $\lambda > 0$ such that $\varepsilon - \lambda/M \geq \varepsilon/2$ and $q_i - \varphi^i (p + \lambda) < \varepsilon/2$. Thus it must be that

\[
\frac{3}{2} \varepsilon \left( F (q_i + \varepsilon + Q_{-i}) - F (q_i + Q_{-i}) \right) \geq \frac{\mu}{M} \left( F \left( q_i + \frac{1}{2} \varepsilon - \frac{Q_{-i}}{2} \right) - F \left( q_i + Q_{-i} \right) \right)
\]

\[
\Leftrightarrow F (q_i + \varepsilon + Q_{-i}) - F (q_i + Q_{-i}) \geq \frac{\mu}{3M} \left[ F \left( q_i + \frac{1}{2} \varepsilon - \frac{Q_{-i}}{2} \right) - F \left( q_i + Q_{-i} \right) \right].
\]

This must hold for all $\varepsilon > 0$. Because $q_i + Q_{-i} < Q$, supply distribution $F$ is Lebesgue absolutely continuous near $q_i + Q_{-i}$; taking the limit as $\varepsilon \searrow 0$ gives

\[
0 \geq \frac{\mu f (q_i + Q_{-i})}{3M}.
\]

Since $f(\cdot) > 0$ at $q_i + Q_{-i}$, this is a contradiction since $M$ is finite (Lemma 10). In this case, bidder $i$ has a profitable deviation. \qed

**Corollary 10.** In any pure-strategy equilibrium, bid functions are strictly decreasing.

We define the derivative of $G^i_b$ with respect to $b$ as follows. For any $q$ and $b^i$, the mapping $t \mapsto G^i(q; b^i + t)$ is weakly decreasing in $t$, and hence differentiable almost everywhere. With some abuse of notation, whenever it exists we denote the derivative of this mapping with respect to $t$ by $G^i_b(q; b^i)$.

**Lemma 8.** For each agent $i$ and almost every $q$ we have:

\[
G^i_b (q; b^i) = f \left( q + \sum_{j \neq i} \varphi^j (b^i (q)) \right) \sum_{j \neq i} \varphi^j_p (b^i (q)) .
\]
Proof. By definition, \( G_i(q; b^i) = \Pr(q^i \leq q|b^i) \). From market clearing, this is
\[
G_i(q; b^i) = \Pr \left( Q \leq q + \sum_{j \neq i} \varphi^j \left( b^i(q) \right) \right) = F \left( q + \sum_{j \neq i} \varphi^j \left( b^i(q) \right) \right).
\]

Where the demands \( \varphi^j \) of agents \( j \neq i \) are differentiable, we have
\[
G_b(q; b^i) = \frac{d}{dq} \left( q + \sum_{j \neq i} \varphi^j \left( b^i(q) \right) \right) \sum_{j \neq i} \varphi^j \left( b^i(q) \right).
\]

Since for all \( j \), the demand function \( \varphi^j \) must be differentiable almost everywhere, the result follows. \( \square \)

Lemma 9. At points where \( G_b(q; b^i) \) is well-defined, the first-order conditions of the pay-as-bid auction are given by
\[
- \left( v(q) - b^i(q) \right) G_b(q; b^i) = 1 - G_i(q; b^i).
\]

In the case of pure strategies under symmetric bidder information, the first-order condition can be written as
\[
- \left( v(q) - b^i(q) \right) \left( \frac{d}{db} Q \left( b^i(q) \right) - \varphi^i \left( b^i(q) \right) \right) = H \left( Q \left( b^i(q) \right) \right),
\]
where \( Q(p) \) is the inverse of \( p(Q) \).

Proof. The agent’s maximization problem is given by
\[
\max_b \int_0^{\overline{Q}} \int_0^q v(x) - b(x) \, dx \, dG_i(q; b).
\]

Integrating by parts, we have
\[
\max_b \left[ \int_0^{\overline{Q}} \int_0^q v(x) - b(x) \, dx \, dG_i(q; b) \right]_{q=0}^{\overline{Q}} + \int_0^{\overline{Q}} \left( v(q) - b(q) \right) \left( 1 - G_i(q; b) \right) \, dq.
\]

In the first square bracket term, both multiplicands are bounded for \( q \in [0, \overline{Q}] \), hence the
fact that $1 - G^i(Q; b) = 0$ for all $b$ and $\int_0^b v(x) - b(x) \, dx = 0$ for all $b$ allows us to reduce the agent’s optimization problem to

$$\max_b \int_0^Q (v(q) - b(q)) \left(1 - G^i(q; b)\right) \, dq.$$ 

The calculus of variations gives us the necessary condition

$$- \left(1 - G^i(q; b^i)\right) - \left(v(q) - b^i(q)\right) G^i_b(q; b^i) = 0.$$ 

This holds at almost all points at which $G^i_b$ is well-defined. Rearrangement yields the first expression for the first-order condition.

To derive the second expression, let us substitute into the above formula for $G^i$ and $G^i_b$ from the Lemma 8. We obtain

$$- \left(v(q) - b^i(q)\right) f \left(q + \sum_{j \neq i} \varphi^j \left(b^i(q)\right)\right) \left(\sum_{j \neq i} \varphi^j_p \left(b^i(q)\right)\right) = 1 - F \left(q + \sum_{j \neq i} \varphi^j \left(b^i(q)\right)\right),$$

Now, $Q(p)$ is well-defined since we have shown that $p$ is strictly monotone. By Corollary 10 bids are strictly monotone in quantities and hence $q + \sum_{j \neq i} \varphi^j \left(b^i(q)\right) = Q(b^i(q))$, and

$$- \left(v(q) - b^i(q)\right) \left(\sum_{j \neq i} \varphi^j_p \left(b^i(q)\right)\right) = H \left(Q(b^i(q))\right).$$

Since $\sum_{j \neq i} \varphi^j_p \left(b^i(q)\right) = \frac{d}{dp} Q(b^i(q)) - \varphi^i_p \left(b^i(q)\right)$, the second expression for the first order condition obtains. \hfill \Box

**Lemma 10.** Each bidder’s equilibrium inverse bid is Lipschitz continuous at all prices $p$ at which the bidder receives a quantity in $[0, q^i(\bar{Q})]$.

**Proof.** Consider an equilibrium bid profile $(b^i)_{i=1}^n$, and let $q^i(Q)$ be the resulting allocation of bidder $i$ given supply $Q$. By way of contradiction, assume that bidder $i$’s inverse bid $\varphi^i$ is not Lipschitz continuous at some price $p$ at which the bidder receives a quantity $q = \varphi^i(p)$ in $[0, q^i(\bar{Q})]$. Then $p = b^i(q)$ and $G^i(q; b^i) < 1$. Let $Q_{\min} \in [0, \bar{Q})$ be a supply at which $q = q^i(Q_{\min})$; in particular, $Q_{\min} = q + \sum_{j \neq i} \varphi^j(b^i(q))$.

The failure of Lipschitz continuity implies that either for any $\bar{K}$ there are arbitrarily small $\varepsilon > 0$ such that $\varphi^i(p - \varepsilon) - \varphi^i(p) > \bar{K}\varepsilon$, or for any $\bar{K}$ there are arbitrarily small $\varepsilon > 0$ such that $\varphi^i(p) - \varphi^i(p + \varepsilon) > \bar{K}\varepsilon$. We provide the argument for the former case; the analysis of the latter cases is analogous.\textsuperscript{82} In this case, for any $K > 0$, there are arbitrarily

\textsuperscript{82}In the former case we maintain the assumption that $b^i$ is right continuous. In the latter case, we
small $\varepsilon > 0$ such that
\[ b^i(q) - b^i(q + \varepsilon) < K\varepsilon. \] (6)

We proceed in five steps. First, we show that bidder $i$ wins an arbitrarily large fraction of residual market quantity just above $Q$. Second, there exist non-trivial intervals on which bidder $i$ wins an arbitrarily large fraction of the residual market quantity. Third, the bid of bidder $i$ is nearly flat on non-trivial intervals just above $Q$. Fourth, each opponent $j$'s bid must be steep near $q^i(Q^\text{min})$. Fifth and finally, the last two claims allow us to conclude that bidder $i$'s inverse bid must be discontinuous at $p$, contradicting Corollary 10 in which we showed that equilibrium bids are strictly decreasing.

**Claim 1.** There is a subsequence of aggregate quantities converging to $Q^\text{min}$ on which $i$ receives all additional supply beyond $Q^\text{min}$, that is for any $M < 1$ and $\varepsilon > 0$, there is $Q \in (Q^\text{min}, Q^\text{min} + \varepsilon)$ such that $q^i(q) > q + (Q - Q^\text{min})M$.

**Proof.** Take any $\varepsilon > 0$ and consider the deviation $b^\varepsilon$ that “kicks out” the bid function at $q$ for length $\varepsilon$,

\[
 b^\varepsilon(q') = \begin{cases} 
 b^i(q) & \text{if } q' \notin [q, q + \varepsilon], \\
 b^i(q) & \text{if } q' \in [q, q + \varepsilon].
 \end{cases}
\]

This deviation increases payment by at most $\int_q^{q+\varepsilon} b^i(q) - b^i(x) \, dx$ whenever the realized quantity $q' > q$, which occurs with probability $1 - G''(q; b^i) \equiv P$. It also increases the allocation: as in equilibrium the opponents bids are strictly decreasing (by Corollary 10), whenever the allocation of $i$ would have been in the interval $(q, q+\varepsilon)$, the allocation increases by $\min(\varepsilon, Q - Q^\text{min})$. The resulting gain in expected utility attributable to the allocation increase is

\[
 \int_{Q^\text{min}}^{Q^\text{max}} \int_{q^i(Q)}^{\min(\varepsilon, Q - Q^\text{min})} v(x) \, dx \, dF(Q),
\]

where $Q^\text{max} = [q + \varepsilon] + \sum_{j \neq i} \varphi^j(b^i(q + \varepsilon))$. Notice that $Q^\text{max} > Q^\text{min} + \varepsilon$. As $(b^j)^n_{j=1}$ is an equilibrium, the costs of the deviation weakly outweigh the benefits,

\[
 \left[ \int_q^{q+\varepsilon} b^i(q) - b^i(x) \, dx \right] P \geq \int_{Q^\text{min}}^{Q^\text{max}} \int_{q^i(Q)}^{\min(\varepsilon, Q - Q^\text{min})} v(x) \, dx \, dF(Q).
\]

The left-hand side is bounded from above by $[b^i(q) - b^i(q + \varepsilon)]\varepsilon P$, and the right-hand side consider $b^\varepsilon$, the left-continuous modification of $b^i$. Since $b^\varepsilon$ and $b^i$ agree almost everywhere, they yield the same utility for bidder $i$, and any utility-improving deviation from $b^\varepsilon$ is a utility-improving deviation from $b^i$, and vice-versa. As, in the latter case, $\varphi^i$ fails Lipschitz continuity to the right of $p$, we conclude that $b^i$ is left-continuous at $q$, so $b^i$ and $b^\varepsilon$ agree at this point and $\varphi^i$ (the inverse of $b^\varepsilon$) also fails Lipschitz continuity to the right of $p$. We may then derive the same contradiction as in the former case.
is bounded from below by
\[
\int_{Q_{\min}}^{Q_{\max}} \int_{q^i(Q)}^{q^i(Q) + \min(\varepsilon, Q - Q_{\min})} v(x) \, dx \, dF(Q)
\]
\[
\geq \int_{Q_{\min}}^{Q_{\max}} (q + \min(\varepsilon, Q - Q_{\min}) - q^i(Q)) \, dF(Q)
\]
\[
\geq f \, v((q + \min(\varepsilon, Q_{\max} - Q_{\min})) (q + \min(\varepsilon, Q - Q_{\min}) - q^i(Q)) \, dQ,
\]
where \( f > 0 \) is a lower bound on \( f(\cdot) \) on \([Q_{\min}, Q_{\max}]\); such a bound exists because \( f \) is continuous and \( f(\cdot) > 0 \) on \([Q_{\min}, Q_{\max}]\) for small \( \varepsilon \) (as then \( Q_{\max} < Q_{\max} \)).

A necessary condition for the alternate bid \( b^\varepsilon \) to not improve bidder \( i \)'s utility is
\[
\left[ b^i(q) - b^i(q + \varepsilon) \right] \varepsilon P
\]
\[
\geq f \, v((q + \min(\varepsilon, Q_{\max} - Q_{\min})) (q + \min(\varepsilon, Q - Q_{\min}) - q^i(Q)) \, dQ
\]
\[
= f \, v(q + \varepsilon) \int_{Q_{\min}}^{Q_{\max}} (q + \min(\varepsilon, Q - Q_{\min}) - q^i(Q)) \, dQ
\]
Let \( C > 0 \) be such that \( C \leq f \, v(q + \varepsilon) / P \); we then require
\[
b^i(q) - b^i(q + \varepsilon) \geq \frac{C}{\varepsilon} \int_{Q_{\min}}^{Q_{\max}} (q + \min(\varepsilon, Q - Q_{\min}) - q^i(Q)) \, dQ. \tag{7}
\]
Consider any \( M \in (0, 1] \) such that
\[
q^i(Q) \leq q + (Q - Q_{\min})M
\]
for \( Q \in (Q_{\min}, Q_{\max}) \); such an \( M \) trivially exists because this inequality holds for \( M = 1 \).

Note that \( q + \varepsilon = q^i(Q_{\max}) \leq q + (Q_{\max} - Q_{\min})M \) implies that
\[
Q_{\max} \geq Q_{\min} + \frac{1}{M} \varepsilon.
\]
The bounds on $Q^\text{max}$ and $q^i(Q)$ imply that
\[
\int_{Q^\text{min}}^{Q^\text{max}} \left( q + \min \left( \varepsilon, Q - Q^\text{min} \right) - q^i(Q) \right) dQ \\
= \int_{Q^\text{min}}^{Q^\text{min} + \varepsilon} \left( q - q^i(q) + Q - Q^\text{min} \right) dQ + \int_{Q^\text{min} + \varepsilon}^{Q^\text{max}} \left( q - q^i(q) + \varepsilon \right) dQ \\
\geq \int_{Q^\text{min}}^{Q^\text{min} + \varepsilon} \left( -(Q - Q^\text{min}) M + Q - Q^\text{min} \right) dQ \\
= \int_{Q^\text{min}}^{Q^\text{min} + \varepsilon} \left( (1 - M) (Q - Q^\text{min}) \right) dQ = (1 - M) \frac{\varepsilon^2}{2}.
\]

Plugging this into the necessary condition above we transform it to
\[
\frac{d}{dQ} \left( \frac{b^i(q) - b^i(q + \varepsilon)}{\varepsilon} \right) \geq \frac{C}{\varepsilon} \frac{1}{2} \frac{(1 - M) \varepsilon^2}{2} = \frac{C \varepsilon^2}{2} (1 - M)
\]

for all sufficiently small $\varepsilon > 0$ and any $M \in (0, 1]$ such that $q^i(Q) \leq q + (Q - Q^\text{min}) M$ for $Q \in (Q^\text{min}, Q^\text{min} + \varepsilon)$.

The above bound and equation 6 jointly imply that, for any $M < 1$ and $\varepsilon > 0$, there is $Q \in (Q^\text{min}, Q^\text{min} + \varepsilon)$ such that $q^i(Q) > q + (Q - Q^\text{min}) M$. This proves the claim: there are supply realizations arbitrarily close to $Q^\text{min}$ for which agent $i$ wins an arbitrarily large proportion of aggregate quantity above $Q^\text{min}$. QED

**Claim 2.** For any $M < 1$ and any $\varepsilon > 0$ there is an aggregate quantity $Q'$ and a quantity $q' = q^i(Q')$ won by bidder $i$ such that for all $\tilde{Q}' \in (Q', Q' + \varepsilon)$,
\[
q^i(\tilde{Q}') \geq q' + (\tilde{Q}' - Q') M.
\]
Furthermore, $Q'$ can be taken to be arbitrarily close to $Q^\text{min}$.

**Proof.** Because $q^i(\cdot)$ is weakly increasing and $q + (Q - Q^\text{min}) M$ continuous in $Q$, by applying Claim 1 to sufficiently larger $M < 1$, we obtain intervals $(Q', Q' + \varepsilon)$ such that for all $\tilde{Q}' \in (Q', Q' + \varepsilon)$,
\[
q^i(\tilde{Q}') \geq q + (\tilde{Q}' - Q^\text{min}) M
\]
as claimed. QED

**Claim 3.** There is a constant $C > 0$ such that for any $M < 1$ and for any $Q'$ from Claim 2 sufficiently close to $Q^\text{min}$ and for any sufficiently small $\delta > 0$, the bids near $q' = q^i(Q')$ satisfy
\[
b^i(q') - b^i(q' + \delta) \leq C (1 - M) \delta.
\]
Proof. Consider $M$, $\varepsilon$, $Q'$, and $q'$ from Claim 2. For $\delta > 0$ consider a deviation

$$b^\delta(q') = \begin{cases} b^i(q' + \delta) & \text{if } q' \in [q', q' + \delta], \\ b^i(q') & \text{otherwise.} \end{cases}$$

This deviation saves payment $\int_{q'}^{q' + \delta} b^i(x) - b^i(q' + \delta) \, dx$ with probability at least $1 - G^i(q' + \delta)$, and, for $\delta$ sufficiently small, we can bound this probability from below by some constant $P > 0$. In equilibrium the saved payment is weakly lower than the associated gross utility loss from winning fewer units; the latter is bounded above by $v(0) (1 - M) \delta (G^i(q' + \delta) - G^i(q'))$, where $(1 - M) \delta$ is the bound on quantity loss implied by the bound in Claim 2. Thus

$$P \int_{q'}^{q' + \delta} b^i(x) - b^i(q' + \delta) \, dx \leq v(0) (1 - M) \left( G^i(q' + \delta) - G^i(q') \right) \delta.$$

As $b^i$ is weakly decreasing, we can bound the left-hand side integral from below by $\frac{1}{2} \delta (b^i(q' + \frac{1}{2} \delta) - b^i(q' + \delta))$ hence obtaining

$$b^i(q' + \frac{1}{2} \delta) - b^i(q' + \delta) \leq \frac{2v(0)}{P} (1 - M) \left( G^i(q' + \delta) - G^i(q') \right) \delta.$$

Because the density of supply is continuous and bounded away from 0 on relevant supply levels and because bidder $i$ receives at least fraction $M$ of any small increase in aggregate supply above $Q'$, there is some real $\bar{f} > 0$ such that $G^i(q' + \delta) - G^i(q') < \bar{f} \delta$ for sufficiently small $\delta$. In effect,

$$b^i(q' + \frac{1}{2} \delta) - b^i(q' + \delta) \leq \frac{2v(0)}{P} \bar{f} (1 - M) \delta.$$

Because this inequality holds for all $\delta$ arbitrarily small, we may telescope it to obtain

$$\lim_{k \to \infty} b^i(q' + \frac{1}{2^k} \delta) - b^i(q' + \delta) \leq \left( \sum_{k=1,2,...} \frac{1}{2^k} \right) \frac{2v(0)}{P} \bar{f} (1 - M) \delta,$$

where the right-hand summation converges to 2. The claim follows from the right-continuity of $b^i$.\textsuperscript{83} QED

**Claim 4.** The bids of $j \neq i$ are steep near $q^i(Q^\text{min})$. That is, there is a constant $C > 0$ such that for any $M < 1$, any sufficiently small $\varepsilon$, and any $Q'$ from Claim 2 sufficiently close

\textsuperscript{83}Recall that we consider the failure of Lipschitz continuity in which for any $\bar{K}$ there are arbitrarily small $\varepsilon > 0$ such that $\varphi^i(p - \varepsilon) - \varphi^i(p) > \bar{K} \varepsilon$. The argument for the failure of Lipschitz continuity in which for any $\bar{K}$ there are arbitrarily small $\varepsilon > 0$ such that $\varphi^i(p) - \varphi^i(p + \varepsilon) > \bar{K} \varepsilon$ needs an adjustment at this point: as mentioned above, in the latter argument we replace $b^i$ with its left-continuous modification $\hat{b}^i$. We then bound $\lim_{k \to \infty} \hat{b}^i(q' - \delta) - \hat{b}^i(q' - \frac{1}{2^k} \delta)$ from above, and the proof proceeds with minimal further changes.
to $Q^{\text{min}}$, the bids near $q_j = q^j (Q')$ satisfy

$$b^j (q_j) - b^j (q_j + \varepsilon) \geq \left[ \frac{M}{1 - M} \right] C \varepsilon.$$

**Proof.** Let $q' = q^i (Q'), M$, and $\delta$ be as in Claim 3 above and $q_j = q^j (Q') = \varphi^j (b^j (q'))$ and note that when $Q'$ is close to $Q^{\text{min}}$ then $q'$ is close to $q = q^i (Q^{\text{min}})$ and $q_j$ is close to $q^j (Q^{\text{min}})$. Let $\varepsilon > 0$ and, for bidder $j \neq i$, consider the deviation $b^\varepsilon$ given by

$$b^\varepsilon (q) = \begin{cases} b^j (q') & \text{if } q \in [q_j, q_j + \varepsilon], \\ b^j (q) & \text{otherwise.} \end{cases}$$

The costs and benefits of this deviation are analogous to those calculated in the proof of Claim 1 for bidder $i$. As the deviation is not profitable in equilibrium, we infer that

$$\left[ \int_{q_j}^{q_j + \varepsilon} b^j (q_j) - b^j (x) \, dx \right] P \geq \int_{Q^{\text{min}}}^{Q^{\text{max}}} \int_{q^j (Q)}^{q^{\text{new}} (Q)} v (x) \, dx \, dF (Q)$$

where $q^{\text{new}} (Q)$ is the allocation of $j$ after the deviation. From Lemma 6 we know that $v(q_j) > b^j (q_j)$; since $F(\cdot) \geq f$, this inequality implies

$$\int_{q_j}^{q_j + \varepsilon} b^j (q_j) - b^j (x) \, dx \geq C_j \int_{Q^{\text{min}}}^{Q^{\text{max}}} q^{\text{new}} (Q) - q^j (Q) \, dQ,$$

for some constant $C_j > 0$ that depends on neither $q_j$ nor $\varepsilon$. The left-hand side can be bounded above,

$$\int_{q_j}^{q_j + \varepsilon} b^j (q_j) - b^j (x) \, dx \leq \left( b^j (q_j) - b^j (q_j + \varepsilon) \right) \varepsilon.$$

By Claim 2 and market clearing, we know that $q^j (Q) \leq q_j + (1 - M) (Q - Q^{\text{min}})$ and hence $Q^{\text{max}} - Q^{\text{min}} \geq \varepsilon / (1 - M)$. As in the analysis of Claim 1, $q^{\text{new}} (Q) = \min \{ q_j + \varepsilon, q_j + Q - Q^{\text{min}} \}$. Since $q^{\text{new}} (Q^{\text{max}}) - q^j (Q^{\text{max}}) = 0$, we have

$$C_j \int_{Q^{\text{min}}}^{Q^{\text{max}}} q^{\text{new}} - q^j (Q) \, dQ \geq C_j \int_{Q^{\text{min}}}^{Q} (Q - Q^{\text{min}}) M \, dQ + C_j \int_{Q}^{Q^{\perp}} \varepsilon - (1 - M) (Q - Q^{\text{min}}) \, dQ,$$

where $Q^{\perp}$ is such that $\varepsilon - (1 - M) (Q^{\perp} - Q^{\text{min}}) = 0$ and $\hat{Q} = Q^{\text{min}} + \varepsilon$; we can truncate the integration at $Q^{\perp}$ because deviation $b^\varepsilon$ weakly increases the quantity allocated to bidder $j$ and hence $q^{\text{new}} (Q) \geq q^j (Q)$ for all $Q$. The right-hand side integrals are $\int_{Q^{\text{min}}}^{\hat{Q}} (Q - Q^{\text{min}}) M \, dQ = \int_{Q^{\text{min}}}^{\hat{Q}} (Q - Q^{\text{min}}) M \, dQ = \cdots$
\[ \frac{1}{2} M \varepsilon^2 \text{ and} \]
\[ \int_{Q}^{Q_{\perp}} \varepsilon - (1 - M) (Q - Q_{\min}) \, dQ = \frac{1}{2} \left[ \varepsilon - (1 - M) \left( \tilde{Q} - Q_{\min} \right) \right] \left[ Q_{\perp} - Q_{\min} \right] \]
\[ = \frac{1}{2} M \varepsilon \left[ \frac{\varepsilon}{1 - M} \right], \]

where the last equation follows from the just-above definitions of \( \tilde{Q} \) and \( Q_{\perp} \). Putting this all together, we have

\[ C_j \int_{Q_{\min}}^{Q_{\max}} q_{\text{new}} - q^i (Q) \, dQ \geq \frac{1}{2} C_j M \varepsilon^2 + \frac{1}{2} C_j M \varepsilon \left[ \frac{\varepsilon}{1 - M} \right] = \frac{1}{2} C_j M \left[ \frac{2 - M}{1 - M} \right] \varepsilon^2. \]

Thus a necessary condition for the deviation not to be profitable is

\[ b^i (q_j) - b^i (q_j + \varepsilon) \geq \frac{1}{2} C_j M \left[ \frac{2 - M}{1 - M} \right] \varepsilon. \]

Because the right-hand side is positive and \( 2 - M > 1 \), the claim obtains for \( C = \frac{1}{2} C_j \). QED

Knowing that the bids of opponents \( j \neq i \) are steep when the bid of bidder \( i \) is flat—and in particular establishing bounds for steepness and flatness in terms of common \( M \)—permits a tighter bound on the quantity lost by a downward deviation for bidder \( i \). Retain \( q_i, M, \) and \( \delta \) as above, let \( \varepsilon > 0 \) and consider a deviation \( b^\varepsilon \),

\[ b^\varepsilon (q) = \begin{cases} 
 b^i (q_i) - \varepsilon & \text{if } b^i (q) \in [b^i (q_i) - \varepsilon, b^i (q_i)], \\
 b^i (q) & \text{otherwise.}
\end{cases} \]

The cost savings of this deviation are bounded below by \( P \int_{q_i}^{\varphi^i(b^i(q_i) - \varepsilon)} b^i (q) - b^\varepsilon (q) \, dq \), where \( P \) is as in Claim 1. This bound is approximated from below by

\[ P \int_{q_i}^{\varphi^i(b^i(q_i) - \varepsilon)} b^i (q) - b^\varepsilon (q) \, dq \geq \frac{1}{2} \left( \varphi^i \left( p - \frac{1}{2} \varepsilon \right) - \varphi^i (p) \right) P \varepsilon. \]

The gross utility sacrificed is bounded above by

\[ \mu T \int_{Q_{\min}}^{\tilde{Q}} Q - Q_{\min} \, dQ + \mu T \int_{Q_{\min}}^{Q_{\max}} \frac{2(n - 1) (1 - M)}{CM (2 - M)} \varepsilon \, dQ, \]

where \( C \) is as in Claim 3. The former term is the quantity lost that results in allocation \( q' = q_i \) (but would have resulted in allocation \( q^i (Q) > q_i \)); the lost quantity in this interval is bounded above by \( Q - Q_{\min} \). The latter term is the quantity lost that results in allocation \( q' > q_i \); the quantity lost in this interval is bounded above by the inverse slope of opponent
bids, established above. Noting that $2 - M \geq 1$, the gross utility sacrificed is bounded by

$$
\left[(\hat{Q} - Q^{\text{min}})^2 + \left(\frac{1 - M}{M}\right) (Q^{\text{max}} - \hat{Q}) (n-1) \right] \frac{2C^{-1}}{\epsilon} \mu \tilde{J} \\
\leq \left[\left(\frac{1 - M}{M}\right) (n-1) 2C^{-1}\right] \frac{2^{-1}}{\epsilon^2} + \left(\frac{1 - M}{M}\right) (Q^{\text{max}} - \hat{Q}) (n-1) \frac{2C^{-1}}{\epsilon} \mu \tilde{J}.
$$

Note that $Q^{\text{max}} - \hat{Q} \leq (\phi'(b^i(q_i) - \epsilon) - q_i)/M$. Substituting through, a necessary inequality is

$$
\frac{1}{2} \left(\phi^i \left(p - \frac{1}{2}\epsilon\right) - \phi^i (p)\right) P \\
\leq \left[\left(\frac{1 - M}{M}\right) (n-1) 2C^{-1}\epsilon\right] \frac{1}{M} (\phi^i (p - \epsilon) - \phi^i (p)) \left[\left(\frac{1 - M}{M}\right) (n-1) 2C^{-1}\right] \mu \tilde{J}.
$$

To economize notation we let $\hat{K} = 1 - M$ and consolidate constants into $C_1$ and $C_2$ (in which we rely on $M$ being close to 1 and thus bound $M^{-1}$ above by 2), thus transforming the above into

$$
\phi^i \left(p - \frac{1}{2}\epsilon\right) - \phi^i (p) \leq \left[C_1 \hat{K} \epsilon + (\phi^i (p - \epsilon) - \phi^i (p)) C_2 \right] \hat{K}.
$$

This gives

$$
(\phi^i (p - \epsilon) - \phi^i (p)) C_2 \hat{K} \geq \phi^i \left(p - \frac{1}{2}\epsilon\right) - \phi^i (p) - C_1 \hat{K}^2 \epsilon.
$$

Because the same inequality must hold for all $\epsilon' \in (0, \epsilon)$, telescoping this inequality implies that for any $k$,

$$
(\phi^i (p - \epsilon) - \phi^i (p)) C_2 \hat{K} \geq \left[\frac{1}{C_2 \hat{K}}\right]^k \phi^i \left(p - \frac{1}{2k+1}\epsilon\right) - \phi^i (p) - \frac{1}{2^k} \left[1 - \left(\frac{2C_2 \hat{K}}{1 - 2C_2 \hat{K}}\right)^{k+1}\right] C_1 \hat{K}^2 \epsilon.
$$

Since $\phi^i$ is not Lipschitz continuous at $p$, for any $K > 0$ and any $k \in \mathbb{N}$ we can find $\epsilon' > 0$ such that $\epsilon' \leq \epsilon/2^k$ and $\phi^i (p - \epsilon') - \phi^i (p) > K \epsilon'$. For such $K$ and $\epsilon'$, let $\tilde{k} = \max\{k: \epsilon' < \epsilon/2^k\}$; by construction, $\epsilon/2 < 2^{k'} \epsilon' \leq \epsilon$. Substituting into the previous inequality gives

$$
(\phi^i \left(p - 2^k\epsilon'\right) - \phi^i (p)) C_2 \hat{K} \geq \left[\frac{1}{C_2 \hat{K}}\right]^\tilde{k} K \epsilon' - \left[1 - \left(\frac{2C_2 \hat{K}}{1 - 2C_2 \hat{K}}\right)^{k+1}\right] C_1 \hat{K}^2 \epsilon' \\
\geq \left[\frac{1}{C_2 \hat{K}}\right]^\tilde{k} K \epsilon' - 2C_1 \hat{K}^2 \epsilon' = \left[\frac{K - 2 \left(\frac{C_2 \hat{K}}{k}\right)^{k} C_1 \hat{K}^2}{\left(\frac{C_2 \hat{K}}{k}\right)^{k}}\right] \epsilon'.
$$
The middle inequality follows from the fact that \( \hat{K} \) may be arbitrarily close to 0, thus
\[
[1 - (2C_2\hat{K})^{k+1}]/[1 - 2C_2\hat{K}] \leq 2
\]
without loss of generality. Similarly, the right-hand term in the numerator is vanishingly small in comparison to the left-hand term (which is independent of \( \bar{k} \)), hence
\[
\varphi_i(p - 2\varepsilon') - \varphi_i(p) \geq \frac{1}{2} \left[ \frac{K}{(2C_2\hat{K})^{k+1}} \right] \varepsilon'.
\]
Recalling that \( \varepsilon/2 < 2\hat{k}\varepsilon' \leq \varepsilon \), we substitute into the previous inequality to obtain
\[
\varphi_i(p - \varepsilon) - \varphi_i(p) \geq \frac{1}{2} \left[ \frac{K\varepsilon}{(2C_2\hat{K})^{k+1}} \right].
\]
Since \( C_2 \) is constant and independent of \( \varepsilon \), and \( \hat{K} \) is arbitrarily close to zero, the fact that \( \hat{k} \) may be arbitrarily large implies that \( \varphi_i(p - \varepsilon) - \varphi_i(p) \geq K'\varepsilon \) for all \( K' \in \mathbb{R} \), contradicting the fact that \( \varphi_i \) is bounded. It follows that \( \varphi_i \) must be Lipschitz continuous at \( p \).

**Lemma 11.** Equilibrium inverse bids are continuously differentiable at all prices \( p \in (p, \bar{p}] \).

**Proof.** Lemma 10 gives that equilibrium inverse bids are Lipschitz continuous. Note that \( G_b^i \) is continuous at a point if the equilibrium first-order conditions are satisfied at this point; let \( Z \) be the set of quantities at which the equilibrium first-order conditions are satisfied. Because the first-order condition is satisfied almost everywhere (Lemma 9), it follows that \( Z \) has full measure and \( G_b^i \) is continuous almost everywhere (Lemma 8). Expressed in terms of inverse bid functions, the first order condition is
\[
\left( v \left( \varphi_i^j (b) \right) - b \right) G_b^i \left( \varphi_i^j (b) ; b \right) = 1 - G_d^j \left( \varphi_i^j (b) ; b \right) = 1 - F \left( \sum_{j=1}^{n} \varphi_j^i (b) \right),
\]
and, because the marginal value \( v \) and all inverse bids \( \varphi_i^j \) are continuous, it follows that there exists a continuous function \( \hat{G}_b^i \) that equals \( G_b^i \) on \( Z \). Because each \( \varphi_i^j \) is monotone it is differentiable on a set \( Z' \) with full measure. Thus on \( Z \cap Z' \), we have
\[
\varphi_{p}^i(p) = \frac{1}{n-1} \sum_{j \neq i} G_b^i \left( \sum_{k} \varphi_k^i (p) \right) - \frac{n-2}{n-1} G_b^i \left( \sum_{k} \varphi_k^i (p) \right).
\]
It follows that there is a function \( \hat{\varphi}_p^i \), continuous on all of \( (p, \bar{p}] \), such that \( \varphi_{p}^i \) equals \( \hat{\varphi}_p^i \) on \( Z \cap Z' \). Since \( \varphi' \) is Lipschitz continuous it is the integral of \( \varphi_p^i \), and since \( \varphi_{p}^i = \hat{\varphi}_p^i|_{Z \cap Z'} \), it is the case that \( \varphi_i^j(p) = -\int_{p}^{\bar{p}} \hat{\varphi}_p^i(x) \, dx \). Since \( \hat{\varphi}_p^i \) is continuous, the fundamental theorem of
calculus implies $\varphi^i_p = \check{\varphi}^i_p$, and the result is shown. \[\square\]

**Corollary 11.** In any equilibrium of the pay-as-bid auction, for all bidders $i$ and for all $q \in [0, \overline{Q}/n)$,

$$-(v(q) - b(q))G^i_b\left(q; b^i\right) = 1 - G^i\left(q; b^i\right).$$

**Lemma 12.** Equilibrium bidding strategies must be symmetric in all pure strategy equilibria: $b^i = b$ for all $i$.

*Proof.* The proof proceeds by establishing an ordering of asymmetric bid functions. We use this ordering to show that equilibrium is symmetric in the $n = 2$ bidder case, and the result from the $n = 2$ bidder case provides tools for the general analysis. Intuitively, the argument is that agents would rather receive a positive quantity rather than zero quantity; because, as we prove, receiving zero quantities is a necessary feature of asymmetric putative equilibria, the asymmetric bids are not best responses. Our proof relies on Lemma 10, which establishes Lipschitz continuity of equilibrium inverse bids; the fundamental theorem of calculus applies, and we have that for any internal price $p$, $\varphi^i(p) = \int_p^P \phi^i(x)dx$.

Note that for any agent $i$, $\sum_{j \neq i} \varphi^j_p(p) = \overline{Q}_p(p) - \varphi^i_p(p)$. Then we can write the agent’s first-order condition as

$$b^i(q) = v(q) + \left(1 - \frac{1}{f(Q(p))}\right)\left(1 - \frac{1}{\overline{Q}_p(p) - \varphi^i_p(p)}\right).$$

Now suppose that two agents $i, j$ have bid functions which differ on a set of positive measure; let $q$ be such that $b^i(q) > b^j(q)$. Then there is a price $p$ such that $\varphi^i(p) > \varphi^j(p)$, and $v(\varphi^i(p)) < v(\varphi^j(p))$. For any such price, substituting into the agents’ first-order conditions gives

$$\left(1 - \frac{1}{f(Q(p))}\right)\left(1 - \frac{1}{\overline{Q}_p(p) - \varphi^i_p(p)}\right) > \left(1 - \frac{1}{f(Q(p))}\right)\left(1 - \frac{1}{\overline{Q}_p(p) - \varphi^j_p(p)}\right).$$

As $1 - F(Q(p)) \neq 0$ (because the inequality is strict), rearrangement gives

$$\varphi^j_p(p) < \varphi^i_p(p).$$

Thus, whenever $\varphi^i(p) > \varphi^j(p)$, we have $\varphi^j_p(p) > \varphi^i_p(p)$. Recalling from Corollary 11 that bids must equal values at $q = \overline{Q}/n$, this implies that if there is any $p$ such that $\varphi^i(p) > \varphi^j(p)$, then $\varphi^i > \varphi^j$.

Now consider the implications for the $n = 2$ bidder case, and let $j \neq i$. Assume that there is $p$ with $\varphi^i(p) > \varphi^j(p) > 0$. Then there is some $\tilde{p}$ such that $\varphi^j(\tilde{p}) = 0$ and $\varphi^i(\tilde{p}) > 0$. \[\text{69}\]
Basic auction logic dictates that bidder $i$ can never outbid the maximum bid of bidder $j$ (i.e., it must be that $b^i(0) = b^j(0)$) thus it must be that bidder $i$'s first-order condition does not apply for initial units, and she is submitting a flat bid. That is, $b^i(q)_{q \leq \varphi^i(\tilde{p})} = \tilde{p}$. Now let $\varepsilon, \lambda > 0$, and define a deviation $\hat{b}^{\varepsilon \lambda}$ for bidder 2,

$$
\hat{b}^{\varepsilon \lambda}(q) = \begin{cases} 
  b^j(0) + \lambda & \text{if } q \leq \varepsilon, \\
  b^j(q) & \text{otherwise.}
\end{cases}
$$

Then for all $q \in (0, \varepsilon]$, $\hat{b}^{\varepsilon \lambda}(q) > b^j(q)$, and when the realized quantity is $Q \in (0, \varepsilon]$ bidder $j$ wins the entire supply. To bound the additional utility, we see that for small $\varepsilon > 0$ bidder $j$ gains at least

$$
\int_0^\varepsilon \left( v(x) - b^j(x) \right) dx \left( F \left( \varphi^i(\tilde{p}) \right) - F(\varepsilon) \right).
$$

There is an extra cost paid as well; to bound this cost we will assume that it is paid with probability 1, and this cost is $(b^j(0) + \lambda)\varepsilon - \int_0^\varepsilon b^j(x)dx$. The deviation $\hat{b}^{\varepsilon \lambda}$ is profitable if the ratio of benefits to costs is greater than 1, hence we look at

$$
\lim_{\lambda \searrow 0, \varepsilon \searrow 0} \frac{\int_0^\varepsilon \left( v(x) - b^j(x) \right) dx \left( F \left( \varphi^i(\tilde{p}) \right) - F(\varepsilon) \right)}{(b^j(0) + \lambda)\varepsilon - \int_0^\varepsilon b^j(x)dx} = \lim_{\varepsilon \searrow 0} \frac{\int_0^\varepsilon \left( v(x) - b^j(x) \right) dx \left( F \left( \varphi^i(\tilde{p}) \right) - F(\varepsilon) \right)}{b^j(0)\varepsilon - \int_0^\varepsilon b^j(x)dx}.
$$

The numerator and denominator both go to zero as $\varepsilon \searrow 0$; application of l'Hôpital’s rule gives

$$
= \lim_{\varepsilon \searrow 0} \frac{v(0) - b^j(0)}{0} = +\infty.
$$

Then either the deviation to $\hat{b}^{\varepsilon \lambda}$ is profitable for bidder $j$ (when $|b^j_q(0)| < \infty$), or bidder $i$ may (essentially) costlessly reduce the initial flat of her bid function (when $|b^i_q(0)| = \infty$).\footnote{Implicit here is that $v(0) > b^j(0) = b^i(0)$, which follows from Lemma 6 but in this particular case is trivial: since bidder $i$ is bidding flat to $\varphi^i(\tilde{p})$, if $v(0) = b^i(0)$ she is obtaining zero surplus on a positive measure of initial units. She would rather cut her bid and lose all of these units with some probability, saving payment for higher units and gaining probable gross utility.}

Now consider the case of $n \geq 3$ agents. By the previous arguments we know that for small quantities submitted bid functions can be ranked (as can their inverses), and that at least two agents submit the highest possible bid function. Thus, we focus on two selected
inverse bid functions, defined pointwise,

\[ \varphi^H(p) \equiv \max \left\{ \varphi^i(p) \right\}, \]
\[ \varphi^L(p) \equiv \max \left\{ \varphi^i(p) : \varphi^i(p) < \varphi^H(p) \right\}. \]

For any asymmetric equilibrium, \( \varphi^L \) is well-defined because the analysis above shows that, unless the inverse bid functions \( \varphi^i, \varphi^j \) are the same for all \( p \), then they are different for all \( p \).

Let \( m_H \equiv \#\{i : \varphi^i = \varphi^H\} \) and \( m_L = \#\{i : \varphi^i = \varphi^L\} \) be the numbers of agents submitting each bid. By the above analysis \( m_H \geq 2 \) and \( m_L \geq 1 \); additionally, \( m_H + m_L \leq n \). As before, there is \( \tilde{p} \) such that \( \varphi^L(\tilde{p}) = 0 \), \( \varphi^H(\tilde{p}) > 0 \), and \( \varphi^L(p) > 0 \) for all \( p < \tilde{p} \). Corollary 10 shows that \( \varphi^H \) must be continuous and Lemma 11 implies that \( \varphi^H_p \) is continuous, hence the equilibrium first order conditions imply

\[ \lim_{p \to \tilde{p}} (m_H - 1) \varphi^H_p(p) = \lim_{p \to \tilde{p}} \left[ (m_H - 1) \varphi^H_p(p) + m_L \varphi^L_p(p) \right]. \]

We now show that if \( \lim_{p \to \tilde{p}} \varphi^L_p(p) = 0 \), then a bidder bidding \( b^L \) has a profitable deviation. Let \( \varepsilon > 0 \) be small, and consider a deviation \( \hat{b}^L \) from \( b^L \) such that

\[ \hat{b}^L(q) = \begin{cases} b^L(\varepsilon) & \text{if } q \leq \varepsilon, \\ b^L(q) & \text{otherwise.} \end{cases} \]

The deviation \( \hat{b}^L \) yields a reduction in quantity bounded above by \( \varepsilon \), at a margin bounded above by \( v(0) \). Because \( \varphi^L_p < \varphi^H \leq 0 \), the probability of reduced quantity is bounded above by \( (m_H + m_L) \overline{f} \varepsilon \), where \( \overline{f} \) is an upper bound for \( f(\cdot) \) in a neighborhood of \( m_H \varphi^H(b^L(0)) \).

The expected gross utility loss from the deviation \( \hat{b}^L \) is therefore bounded above by \( (m_H + m_L) \overline{f} v(0) \varepsilon^2 \). On the other hand, the deviation \( \hat{b}^L \) saves the bidder payment for all quantity realizations \( q > \varepsilon \). This payment is saved with probability bounded below by some \( P > 0 \), and, because \( \varphi^L(\tilde{p}) = 0 \) and \( \lim_{p \to \tilde{p}} \varphi^L_p(p) = 0 \), for any \( C > 0 \) there is sufficiently small \( \varepsilon \) such that the amount saved bounded from below by \( \varepsilon^2/C \). The deviation is profitable if

\[ (m_H + m_L) \overline{f} v(0) \varepsilon^2 < \frac{\varepsilon^2}{C}. \]

After factoring out the common \( \varepsilon^2 \) term, the left-hand side is constant while the right-hand side can be made arbitrarily large for sufficiently small \( \varepsilon \). It follows that \( \hat{b}^L \) is a profitable deviation for some \( \varepsilon \).
Then it cannot be the case that \( \lim_{p \to \tilde{p}} \varphi^L_p (p) = 0 \). It follows that

\[
\lim_{p \to \bar{p}} \varphi^H_p (p) = \lim_{p \to \bar{p}} \varphi^H_p (p) + \frac{m_L}{m_H - 1} \varphi^L_p (p) < 0.
\]

Intuitively, the bid function \( b^H \) is steeper below \( \varphi^H (\tilde{p}) \) than above, and there is a kink at this point. This implies a discontinuity in a bidder \( L \)'s first-order condition near \( q = 0 \). For \( p \) close to but less than \( \tilde{p} \), the first-order condition is

\[
- \left( v \left( \varphi^L (p) \right) - p \right) f (Q (p)) \left( m_H \varphi^H_p (p) + (m_L - 1) \varphi^L_p (p) \right) - (1 - F (Q (p))) = 0,
\]

\[
\Rightarrow - \left( v \left( \varphi^L (p) \right) - p \right) f (Q (p)) \left( (m_H - 1) \varphi^H_p (p) + m_L \varphi^L_p (p) \right) - (1 - F (Q (p))) > 0.
\]

Letting \( p \nearrow \tilde{p} \), we know that the term \( [(m_H - 1) \varphi^H_p (p) + m_L \varphi^L_p (p)] \) approaches \( \lim_{p \to \bar{p}} (m_H - 1) \varphi^H_p (p) \), proportional to the marginal probability gained by a slight increase in bid from \( b^L \) near \( \tilde{p} \) to \( \bar{b} > \tilde{p} \). Thus, the \( L \) bidder’s second-order conditions are not satisfied near \( q = 0 \), and this is not an equilibrium.

\[\square\]

C Proofs for Section 3 (Pay-as-Bid Equilibrium)

For our proofs of Theorems 2, 3, and 4, we assume that the reserve price is \( R = 0 \). In this case, the maximum realizable quantity is \( \overline{Q}^R = Q \). In Supplementary Appendix C.5 we detail how these proofs must change to account for binding reserve prices.

C.1 Proof of Theorem 2 (Uniqueness)

Proof. From Lemma 9 and market clearing, we know that for all bidders

\[
(p (Q) - v (q)) G_b (q; b^i) = 1 - G_b (q; b^i).
\]

Since Lemma 12 tells us that agents’ strategies are symmetric, Lemma 8 allows us to write this as

\[
\left( p (Q) - v \left( \frac{1}{n} Q \right) \right) (n - 1) \varphi_p (p (Q)) = H (Q).
\]

From market clearing, we know that \( p (Q) = b (Q / n) \); hence \( p_Q (Q) = b_q (Q / n) / n \). Additionally, standard rules of inverse functions give \( \varphi_p (p (Q)) = 1 / b_q (Q / n) \) almost everywhere. Thus we have

\[
\left( p (Q) - v \left( \frac{1}{n} Q \right) \right) \frac{n - 1}{n} = H (Q) p_Q (Q).
\]
Now suppose that there are two solutions, \( p \) and \( \hat{p} \). From Corollary ?? we know that \( p(\overline{Q}) = \hat{p}(\overline{Q}) \). Suppose that there is a \( Q \) such that \( \hat{p}(Q) > p(Q) \); taking \( Q \) near the supremum of \( Q \) for which this strict inequality obtains we conclude that \( \hat{p}_Q(Q) < p_Q(Q) \).

But then we have

\[
\hat{p}(Q) > p(Q) = v\left(\frac{1}{n}Q\right) + \left(\frac{n}{n-1}\right) H(Q)p_Q(Q) > v\left(\frac{1}{n}Q\right) + \left(\frac{n}{n-1}\right) H(Q)\hat{p}_Q(Q).
\]

The presumed right-continuity of bids and Lipschitz continuity of \( \varphi \) (from Lemma 10) allow us to conclude that if \( p \) solves the first-order conditions, \( \hat{p} \) cannot.

\[\Box\]

### C.2 Proof of Theorem 3 (Bid Representation)

From the first order condition established in the proof of uniqueness, the equilibrium price satisfies

\[
p_Q = p\tilde{H} - \hat{\varphi},
\]

where \( \hat{\varphi}(x) = v(x/n) \), and \( \tilde{H}(x) = [1/H(x)][(n-1)/n] \). The solution to this equation has general form

\[
p(Q) = Ce^{\int_0^Q \tilde{H}(x)dx} - e^{\int_0^Q \tilde{H}(x)dx} \int_0^Q e^{-\int_0^x \tilde{H}(y)dy} \tilde{H}(x) \hat{\varphi}(x) dx,
\]

parametrized by \( C \in \mathbb{R} \). Define \( \rho = \frac{n-1}{n} \in [\frac{1}{2}, 1) \). We can see that \( \tilde{H} = -\rho \frac{d}{dq} \ln(1 - F) \).

Thus we have

\[
e^{\int_0^t \tilde{H}(x)dx} = e^{-\rho \int_0^t \frac{d}{dx} \ln(1 - F(x))dx} = e^{-\rho(\ln(1 - F(t)) - \ln 1)} = (1 - F(t))^{-\rho}.
\]

Substituting and canceling, we have for \( Q < \overline{Q} \):

\[
p(Q) = \left(C - \rho \int_0^Q f(x) (1 - F(x))^{\rho-1} \hat{\varphi}(x) dx\right) (1 - F(Q))^{-\rho}.
\]

\[\text{85}\] The inequality inversion here from usual derivative-based approaches reflects the fact that we are “working backward” from \( \overline{Q} \), while any solution must be weakly decreasing: thus a small reduction in \( Q \) should yield \( \hat{p}(\overline{Q}) = p(\overline{Q}) \leq p < \hat{p} \).

\[\text{86}\] The first-order condition for bids ensures that the slope of \( \varphi \) is strictly negative; thus since \( \varphi \) is Lipschitz continuous (by Lemma 10) any equilibrium inverse bid is the integral of its own derivative, and any equilibrium market price function is the integral of its own derivative.
Since $1 - F(\bar{Q}) = 0$, this implies that $C = \rho \int_{0}^{\bar{Q}} f(x) (1 - F(x))^{\rho - 1} \hat{v}(x) \, dx$. The market clearing price is then given by

$$p(\bar{Q}) = \rho \int_{0}^{\bar{Q}} f(x) (1 - F(x))^{\rho - 1} \hat{v}(x) \, dx (1 - F(\bar{Q}))^{-\rho}.$$ 

Since $d/dy[F^{\alpha}(y)] = \rho f(y)(1 - F(y))^{\rho - 1}(1 - F(Q))^{-\rho}$, our formula for market price obtains, and since we have proven earlier that the equilibrium bids are symmetric, the formula for bids obtains as well.

### C.3 An Alternative Form of the Existence Condition, and Proofs of Theorem 4 (Existence) and Corollary 1

We can weaken the global maximum assumption in Theorem 4 to the assumption that at almost every $q \in [0, \bar{Q}^R/n]$, the first derivative of $U(\cdot; q)$ is zero only at $\hat{q}$ at which $U(\cdot; q)$ attains the global maximum on $\hat{q} \in [0, \bar{Q}^R/n] \cap \{q : b(\hat{q}) = v(q)\}$. Equivalently, at almost every $q \in [0, \bar{Q}^R/n]$, $(v(q) - b)(1 - F(q + (n - 1)\varphi(b)))$ attains its global maximum on $b \in [p(\bar{Q}^R), v(q)] \cap [p(\bar{Q}^R), b(0)]$ at $b(\hat{q})$. These weaker conditions follow from the argument below when we recognize that any bid function that takes values $b(q) > v(q)$ on some $q \in [0, \bar{Q}^R/n]$ is dominated by a bid function that are never above $v(q)$ on $q \in [0, \bar{Q}^R/n]$.

**Proof of Theorem 4.** The proof of equilibrium existence under deterministic supply is given in the main text, therefore we assume in this proof that supply has full support, Supp $Q = [0, \bar{Q}]$. Let us fixed a bidder $i$ whose incentives we will analyze, and assume that other bidders $j \neq i$ follow the strategies $b^j = b$ of Theorem 3 when bidding on quantities $q \leq \bar{Q}^R/n$, and that they bid $b^j(\bar{Q}^R/n) = v(\bar{Q}^R/n)$ for quantities $q \in [\bar{Q}^R/n, \bar{Q}^R/(n - 1)]$. Since bids and values are weakly decreasing, in equilibrium there is no incentive for bidder $i$ to lower or raise their bid on any quantity $q \geq \bar{Q}^R/n$ and we only need to check that bidder $i$ finds it optimal to submit bids prescribed by Theorem 3 for quantities $q \in [0, \bar{Q}^R/n]$.

Because the bid $b$ derived in Theorem 3 is strictly decreasing on $[0, \bar{Q}^R/n]$ and the auction is discriminatory, a bid $\tilde{b}$ such that there is a $q$ with $\tilde{b}(q) > b(0)$ is weakly dominated by a bid which is never above $b(0)$. Second, since opponents’ bid $b$ is never below $v(\bar{Q}^R/n)$ on $[0, \bar{Q}^R/(n - 1)]$, a bid $\tilde{b}$ and quantity $q$ such that $\tilde{b}(q) < v(\bar{Q}^R/n)$ is never awarded quantity $q$. These two facts in turn imply that the bidder’s optimal bid for any quantity is $\tilde{b}(q) \in [v(\bar{Q}^R/n), b(0)]$. Finally, since bid $b$ is continuous and, by Theorem 1, is such that $b(\bar{Q}^R/n) = v(\bar{Q}^R/n)$, it is the case that for any utility-maximizing bid $\tilde{b}$ and any quantity $q$, there is a quantity $\hat{q} \in [0, \bar{Q}^R/n]$ such that $b(q) = b(\hat{q})$. Because $b$ is strictly decreasing on $\hat{q} \in [0, \bar{Q}^R/n]$, the preceding equality defines a unique mapping $\hat{q}$ from $q$ to $\hat{q}$. As shown in
the proof of Lemma 9, bidder \( i \)'s expected utility from submitting bid \( \tilde{b} \) is

\[
\mathbb{E} \left[ u^i (\tilde{b}) \right] = \int_0^{Q} (v(q) - \tilde{b}(q)) \left( 1 - F \left( q + (n - 1) \varphi \circ \tilde{b}(q) \right) \right) dq,
\]

and it follows that we may write the expected utility from bidding \( b \circ \tilde{q} \) as

\[
\mathbb{E} \left[ u^i (b \circ \tilde{q}) \right] = \int_0^{Q} (v(q) - b \circ \tilde{q}(q)) \left( 1 - F \left( q + (n - 1) \tilde{q}(q) \right) \right) dq = \int_0^{Q} U(\tilde{q}(q); q) dq.
\]

In particular, instead of bidder \( i \) selecting a bid for quantity \( q \), we may consider bidder \( i \) as selecting a bid such that their opponents each receive quantity \( \tilde{q}(q) \).

From \( U(\tilde{q}(q); q) \leq \max_{\tilde{q} \in [0, Q^R/n]} U(\tilde{q}; q) \), we then infer that

\[
\mathbb{E} \left[ u^i (\tilde{q}) \right] \leq \int_0^{Q} \max_{\tilde{q} \in [0, 1/n Q^R]} U(\tilde{q}; q) dq.
\]

In particular, any bid which maximizes \( U(\cdot; q) \) pointwise for almost every quantity \( q \) will maximize the bidder's expected utility. As we showed in Appendix B.2, the first derivative of \( U(\cdot; q) \) is the pointwise first-order condition used to derive the bid \( b \), and is equal to zero at \( \tilde{q} = q \). Then by the assumption of this theorem, \( U(\cdot; q) \) is maximized at \( \tilde{q} = q \) for almost every \( q \), and thus \( \hat{b} = b \) is a best response to bidder \( i \)'s opponents submitting the symmetric bid \( b^i = b \).

**Proof of Corollary 1.** Denote by \( \varphi^n \) the equilibrium inverse bid when there are \( n \) bidders. Note that for every \( q \in [0, Q^R/n] \) and \( p \in (v(Q^R/n), v(q)) \), the expression

\[
(v(q) - p) \left( 1 - F \left( q + (n - 1) \varphi^n(p) \right) \right)
\]

is differentiable in \( p \), nonnegative, and has limit 0 as \( p \to v(q) \). To establish the condition in Theorem 4, in its weaker version at the beginning of this appendix, it is thus sufficient to show that, for almost all relevant \( q \), the derivative of this expression with respect to \( p \) is zero at most once.

The derivative is

\[
-(1 - F \left( q + (n - 1) \varphi^n(p) \right)) - (v(q) - p) (n - 1) f \left( q + (n - 1) \varphi^n(p) \right) \varphi^n(p). \tag{9}
\]

From the equilibrium derivation in Theorem 3, this derivative is zero at \( p = b^n(q) \). We now show that when \( n \) is large this derivative is negative for \( p > b^n(q) \) and positive for \( p < b^n(q) \).

Our first step is to show that, under the assumptions of the Corollary the slope of the
We first show that \( v \), the inverse bid, \( \varphi_p^n \), is bounded and bounded away from zero. Because \( \varphi_p^n(p) = 1/b_q^n(\varphi^n(p)) \), it is sufficient to show that the slope of the equilibrium bid, \( b_q^n \), is bounded and bounded away from zero. Integrating our bid representation (1) by parts gives

\[
b^n(q) = v(q) + \int_q^{\overline{q}} v_q(x) (1 - \frac{F^n(q)(x)}{\overline{F^n}(q)}) dx = v(q) + \int_q^{\overline{q}} v_q(x) \left( \frac{1 - F^\mu(x)}{1 - F^\mu(q)} \right)^\frac{n-1}{n} dx.
\]

Then the derivative of the equilibrium bid function is

\[
b_q^n(q) = \frac{n-1}{n} \int_q^{\overline{q}} v_q(x) \left( \frac{1 - F^\mu(x)}{1 - F^\mu(q)} \right)^\frac{n-1}{n} \frac{f^\mu(q)}{1 - F^\mu(q)} dx.
\]

We first show that \( b_q^n \) is bounded away from zero. Recalling that \( b_q^n \leq 0 \), that \( v \leq v_q(x) \leq v < 0 \) by assumption, and that \( 0 < \frac{f^\mu}{\overline{f^\mu}} < f^\mu < \frac{f^\mu}{\overline{f^\mu}} \) by assumption, we have

\[
b_q^n(q) \leq \frac{n-1}{n} \left( \frac{1}{1 - F^\mu(q)} \right)^{\frac{n-1}{n}+1} \int_q^{\overline{q}} v f^\mu \left( 1 - \frac{F^\mu(x)}{1 - F^\mu(q)} \right)^\frac{n-1}{n} dx
\]

\[
\leq \frac{n-1}{n} \left( \frac{1}{1 - F^\mu(q)} \right)^{\frac{n-1}{n}+1} \frac{v f^\mu}{\overline{f^\mu}} \int_q^{\overline{q}} \left( 1 - \frac{F^\mu(x)}{1 - F^\mu(q)} \right)^\frac{n-1}{n} \frac{f^\mu(x)}{\overline{f^\mu}} dx
\]

\[
\leq \frac{n-1}{n} \left( \frac{1}{n+1} \right) \frac{v f^\mu}{\overline{f^\mu}} = \frac{n-1}{2n-1} \left[ \frac{v f^\mu}{\overline{f^\mu}} \right] \leq \frac{1}{3} \left[ \frac{v f^\mu}{\overline{f^\mu}} \right] < 0.
\]

To see that \( b_q^n \) is bounded below follows a similar path,

\[
b_q^n(q) \geq \left( \frac{\overline{f^\mu}}{(\overline{f^\mu} - q) \overline{f^\mu}} \right) \int_q^{\overline{q}} v dx = \left( \frac{\overline{f^\mu}}{(\overline{f^\mu} - q) \overline{f^\mu}} \right) (\overline{f^\mu} - q) v = \frac{\overline{f^\mu} v}{\overline{f^\mu}}.
\]

Then \( b_q^n \), and hence \( \varphi_p^n \), is bounded and bounded away from zero. Note that these bounds are independent of the number of bidders \( n \).

Because the density \( f^\mu \) and its derivative \( f_q^\mu \) are bounded, and because \( \varphi_p^n \) is bounded uniformly for all \( n \), we can write (9) as

\[
- \left( 1 - F^\mu \left( \frac{q + (n-1) \varphi^n(p)}{n} \right) \right) - \frac{n-1}{n} (v(q) - p) f^\mu \left( \frac{q + (n-1) \varphi^n(p)}{n} \right) \varphi^n(p)
\]

\[
= - (1 - F^\mu(\varphi^n(p))) - \frac{n-1}{n} (v(q) - p) f^\mu(\varphi^n(p)) \varphi_p^n(p)
\]

\[
- \left( F^\mu(\varphi^n(p)) - F^\mu(\frac{q + (n-1) \varphi^n(p)}{n}) \right) - \frac{n-1}{n} (v(q) - p) \left( f^\mu \left( \frac{q + (n-1) \varphi^n(p)}{n} \right) - f^\mu(\varphi^n(p)) \right)
\]

\[
= - (1 - F^\mu(\varphi^n(p))) - \frac{n-1}{n} (v(q) - p) f^\mu(\varphi^n(p)) \varphi_p^n(p) - \frac{1}{n} (q - \varphi^n(p)) \hat{C},
\]

76
where

\[
\frac{1}{n} (q - \varphi^n(p)) \hat{C}_1 = - \left(F^\mu (\varphi^n(p)) - F^\mu \left(\frac{q + (n-1)\varphi^n(p)}{n}\right)\right) - \frac{n-1}{n} \left(v(q) - p\right) f^\mu \left(\frac{q + (n-1)\varphi^n(p)}{n}\right) \varphi^n(p) \\
= \frac{1}{n} (q - \varphi^n(p)) c_{F^\mu} - \frac{n-1}{n} \left[v(q) - p\right] \varphi^n(p) \left(q - \varphi^n(p)\right) c_{f^\mu} \\
= \frac{1}{n} (q - \varphi^n(p)) c_{F^\mu} - \frac{1}{n} (q - \varphi^n(p)) c_{f^\mu} c_{f^\mu} = \frac{1}{n} (q - \varphi^n(p)) (c_{F^\mu} - c_{f^\mu}).
\]

The constants \(c_{F^\mu}\) and \(c_{f^\mu}\) exist and are bounded, independent of \(p, q,\) and \(n\), because \(f^\mu\) and \(f^\mu_q\) are bounded. The constant \(c_\delta\) is bounded, independent of \(p, q,\) and \(n\), because \(v(q) - p\) and \(\varphi^n(p)\) are bounded, independent of \(n\). It follows that the constant \(\hat{C}_1\) exists and has a uniform bound which is independent of \(p, q,\) and \(n\). From our equilibrium bid representation, we may then write (9) as

\[
\frac{n-1}{n} \left[v(\varphi^n(p)) - p\right] \varphi^n(p) - \frac{n-1}{n} \left[v(q) - p\right] \varphi^n(p) - \frac{1}{n} (q - \varphi^n(p)) \frac{\hat{C}_1}{f^\mu (\varphi^n(p))}.
\]

Since \(f^\mu\) and \(\varphi^n(p)\) are bounded away from zero, (9) has the same sign as

\[
- \left[(v(\varphi^n(p)) - v(q)) - \frac{1}{n} (q - \varphi^n(p)) \hat{C}_2\right],
\]

where \(\hat{C}_2 = \hat{C}_1/\varphi^n(p)\) is bounded because \(\varphi^n(p)\) is bounded away from 0 uniformly for all \(n\). Further, because the derivative of \(v\) is bounded away from zero, there is \(\hat{v} < 0\) such that the derivative we study has the same sign as

\[
- \left[(\varphi^n(p) - q) \hat{v} - \frac{1}{n} (q - \varphi^n(p)) \hat{C}_2\right] = (\varphi^n(p) - q) \left(|\hat{v}| - \frac{1}{n} \hat{C}_2\right).
\]

Although the specific values of \(\hat{v}\) and \(\hat{C}_2\) depend on \(p, q,\) and \(n\), they are nonetheless uniformly bounded. Since \(\hat{v}\) is bounded away from zero, it follows that there is \(n\) sufficiently large so that (9) is negative when \(p > b^n(q)\) and positive when \(p < b^n(q)\), completing the proof.
C.4 Verification of an Existence Example

Linear marginal values with generalized Pareto distribution of supply. For generalized Pareto distributions with parameter $\alpha > 0$,

\[
1 - F(x) = \left(1 - \frac{x}{\overline{Q}}\right)^\alpha, \quad f(x) = \frac{\alpha}{\overline{Q}} \left(1 - \frac{x}{\overline{Q}}\right)^{\alpha - 1};
\]

\[
H(x) = \frac{1}{\alpha} \left(\overline{Q} - x\right), \quad H_q(x) = - \frac{1}{\alpha}.
\]

Then with linear market values $v(q) = \beta_0 - q \beta_q$,

\[
- \frac{1}{\alpha} \left(\overline{Q} - n \varphi(p)\right) \beta_q + \frac{1}{\alpha} \left(\beta_0 - \varphi(p) \beta_q - p\right) \propto \beta_0 - \left(\overline{Q} - (n - 1) \varphi(p)\right) \beta_q - p.
\]

For all $Q < \overline{Q}$, $p(Q) > p(\overline{Q})$ and $\overline{Q} > n \varphi(p(Q))$; hence for all $Q < \overline{Q}$,

\[
\beta_0 - \left(\overline{Q} - (n - 1) \varphi(p)\right) \beta_q - p < \beta_0 - \frac{1}{n} \overline{Q} \beta_q - p \left(\overline{Q}\right) = 0.
\]

Then the existence condition is satisfied for all $Q \in [0, \overline{Q})$.

C.5 Modifying the Proofs to Allow for Reserve Prices

The bound on market price established in Theorem 1 implies that a binding reserve price is equivalent to creating an atom in the supply distribution at the quantity at which marginal value equals the reserve price. In order to extend the previous proofs to the setting that allows reserve prices (as the results are stated in the main text), we therefore need to extend them to distributions in which there might be an atom at the upper bound of support $\overline{Q}$.\(^{87}\)

All our results remain true, and the proofs go through without much change except for the end of the proof of Theorem 3, where more care is needed.

The proof of Theorem 3 goes through until the claim that $1 - F(\overline{Q}) = 0$; in the presence of an atom at $\overline{Q}$ this claim is no longer valid. We thus proceed as follows. We multiply both sides of equation (8) by $(1 - F(Q))^\rho$ and conclude that

\[
p(Q) (1 - F(Q))^\rho = C - \rho \int_0^Q f(x) (1 - F(x))^{\rho - 1} \hat{v}(x) \, dx.
\]

Now, let $\hat{F}(\overline{Q}) \equiv \lim_{Q \searrow \overline{Q}} F(Q')$. Because the market price and the right-hand integral

\(^{87}\)Starting with a given supply distribution $F$ with support $[0, \overline{Q}]$ and moving all probability from $[\overline{Q}^R, \overline{Q}]$ to an atom at $\overline{Q}^R$ results in a new distribution $\hat{F}$ with support $[0, \overline{Q}^R]$, with an atom at $\overline{Q}^R$. All results apply to this new distribution, thus it is without loss of generality to assume that the mass point is at $\overline{Q}$.
are continuous as $Q \nearrow \bar{Q}$, we have

$$p \left( \bar{Q} \right) \left( 1 - \hat{F} \left( \bar{Q} \right) \right) = C - \rho \int_0^{\bar{Q}} f \left( x \right) \left( 1 - F \left( x \right) \right)^{\rho-1} \hat{v} \left( x \right) dx.$$  

The parameter $C$ is determined by this equation. The market price function is then

$$p \left( Q \right) = \left( \frac{1 - \hat{F} \left( \bar{Q} \right)}{1 - F \left( Q \right)} \right) ^\rho p \left( \bar{Q} \right) + \rho \int_Q^{\bar{Q}} f \left( x \right) \left( 1 - F \left( x \right) \right)^{\rho-1} \hat{v} \left( x \right) dx \left( 1 - F \left( Q \right) \right)^{-\rho}. \quad (10)$$

Recall from Corollary ?? that $p(\bar{Q}) = v(\bar{Q}/n)$. Extending our notation to the auxiliary distribution $F^{Q,n}$, we also have

$$F^{Q,n} (\bar{Q}) - F^{Q,n} (Q) = 1 - F^{Q,n} (\bar{Q}) = \left( \frac{1 - \hat{F} \left( \bar{Q} \right)}{1 - F \left( Q \right)} \right) ^\rho.$$  

Since $d/dy[F^{Q,n}(y)] = \rho f(y)(1 - F(y))^{\rho-1}(1 - F(Q))^{-\rho}$ for all $Q, y < \bar{Q}$, we have

$$p \left( Q \right) = \left( F^{Q,n} (\bar{Q}) - F^{Q,n} (Q) \right) \hat{v} \left( \bar{Q} \right) + \int_Q^{\bar{Q}} \hat{v} \left( x \right) dF^{Q,n} (x),$$  

proving our formula for equilibrium stop-out price in the presence of an atom at $\bar{Q}$. Noting that $\bar{Q}_R < \bar{Q}$ implies an atom in the realized allocation distribution at $\bar{Q}_R$, equation 2 in Theorem 3 follows. Since equilibrium is symmetric, equation 1 is an immediate corollary. □

D Proofs for Section 4 (Designing Pay-as-Bid Auctions): Proof of Theorem 5

Theorem 5 shows that, when the designer is constrained to a reserve price $R$ and a distribution over supply $F$, the optimal mechanism is deterministic. This is distinct and does not follow from the analysis in Appendix A, which shows that (under regularity conditions on demand) a seller who can implement stochastic elastic supply prefers to implement a deterministic elastic supply curve. In general, fixed supply $Q^*$ and reserve $R^*$ is insufficiently elastic to obtain monopoly rents from all bidder signals $s$, and a seller who can implement an elastic supply curve will strictly prefer to do so.
Proof of Theorem 5. Consider a pure-strategy equilibrium in a pay-as-bid auction with reserve price $R$ and supply distribution $F$. In Section 3 we proved that the equilibrium is essentially unique and symmetric. Furthermore, in equilibrium, for any relevant quantity $q$, each bidder’s bid equals the resulting market-clearing price when quantity $Q = nq$ is sold; we denote this market clearing price $p(Q; R, s)$, suppressing in the notation the price’s dependence on $F$ as it is constant. We denote the resulting equilibrium revenue by $\pi(Q; R, s)$ and we write $\hat{v}(y; s) = v(y/n; s)$ for a bidder’s marginal value from his or her share of quantity sold $y$.

The seller maximizes the expected revenue $\mathbb{E}_s [\pi^F] = \mathbb{E}_s \int_0^Q \pi(Q; R, s) dF(Q)$, where $\pi^F$ denotes the seller’s profits when bidders bid against distribution of supply $F$. When bidders’ values are low relative to the reserve price, and the realized quantity is high, the reserve price is binding and the bidders receive only a partial allocation. Expected revenue is

$$\mathbb{E}_s [\pi^F] = \mathbb{E}_s \int_0^Q \int_0^{Q^R(y,s)} p(x; R, s) dx dF(y). \quad (11)$$

Integrating by parts gives

$$\mathbb{E}_s [\pi^F] = \mathbb{E}_s \left\{ \left[ - (1 - F(y)) \int_0^{Q^R(y,s)} p(x; R, s) dx \right] \right|_{y=0}^{Q} + \int_0^Q (1 - F(y)) p(Q^R(y, s); s) dQ^R(y, s) \right\},$$

where the first addend is zero. Recognizing that $Q$ is continuous in $y$ and that $Q^R(y, s) = 1$ for $v(y/n; s) > R$ and $Q^R(y, s) = 0$ for $v(y/n; s) < R$, we can thus express the expected revenue as

$$\mathbb{E}_s [\pi^F] = \mathbb{E}_s \int_0^{Q^R(s)} (1 - F(y)) p(Q^R(y, s); s) dy. \quad (12)$$

Applying our Theorem 3 gives

$$\mathbb{E}_s [\pi^F] = \mathbb{E}_s \int_0^{Q^R(s)} (1 - F(y)) \left[ (1 - F^y,n(Q^R(s))) \hat{v}(Q^R(s); s) + \int_y^{Q^R(s)} \hat{v}(x; s) dF^y,n(x) \right] dy,$$

where $F^y,n(x) = 1 - \left( \frac{1-F(x)}{1-F(y)} \right)^{\frac{n-1}{n}}$ is the c.d.f. of the weighting distribution from the theorem.88

88The outer integral in equation (12) is bounded to $[0, Q^*(s)]$, thus $y \leq Q^*(s)$ for all $y$ and $F^y,n(Q^*(s))$ is well-defined. The left-hand addend in the integral results from the fact that, when $Q^*(s) < Q^*$—that is, when signal-$s$ bidders have low values for the maximum quantity, $\hat{v}(Q^*) < R$—there is a mass point in the resulting distribution of realized aggregate allocation at $Q^*(s)$; this same expression is seen in equation (10) in Appendix (C.5).
Applying integration by parts to the inner integral and substituting in for $F_{y,n}$ gives

$$E_s \left[ \pi F \right] = E_s \int_0^{Q^R(s)} (1 - F(y)) \hat{v}(y; s) + (1 - F(y)) \frac{1}{2} \int_y^{Q^R(s)} \hat{v}_q(x; s) (1 - F(x)) \frac{n-1}{n} \ dx \ dy. \tag{13}$$

We may change the order of integration of the right-hand double integral to obtain

$$\int_0^{Q^R(s)} (1 - F(y)) \frac{1}{2} \int_y^{Q^R(s)} \hat{v}_q(x; s) (1 - F(x)) \frac{n-1}{n} \ dx \ dy$$

$$= \int_0^{Q^R(s)} \int_x^0 (1 - F(y)) \frac{1}{2} \ dy \hat{v}_q(x; s) (1 - F(x)) \frac{n-1}{n} \ dx$$

$$\leq \int_0^{Q^R(s)} x \hat{v}_q(x; s) (1 - F(x)) \ dx,$$

where the inequality follows from the facts that $\hat{v}_q \leq 0$, and $1 - F(y) \geq 1 - F(x)$ for $y \leq x$. Substituting $y$ for $x$ and plugging this bound in the above expression for expected profits, we have

$$E_s \left[ \pi F \right] \leq E_s \int_0^{Q^R(s)} (1 - F(y)) \left( \hat{v}(y; s) + y \hat{v}_q(y; s) \right) dy.$$

Notice that $x \hat{v}_q(x; s) + \hat{v}(x; s) = \pi_\delta(x; s)$, where $\pi_\delta(x; s) = x \hat{v}(x; s)$ is the revenue from selling quantity $x$ at price $\hat{v}(x; s)$. Integrating by parts gives

$$E_s \left[ \pi F \right] \leq E_s \left[ \int_0^{Q^R(s)} \pi_\delta(x; s) (1 - F(x)) \ dx \right]$$

$$= E_s \left[ \int_0^{Q^R(s)} \pi_\delta(Q^R(x,s); s) (1 - F(Q^R(x,s))) + \int_0^{Q^R(s)} \pi_\delta(x; s) F(x) \ dx \right]$$

$$= E_s \left[ \int_0^{Q^R(s)} \pi_\delta(Q^R(x,s); s) F(x) \ dx \right]. \tag{14}$$

Thus,

$$E_s \left[ \pi F \right] \leq \int_0^{Q^R(s)} E_s \left[ \pi_\delta(Q^R(x,s); s) \right] F(x).$$

Since there are no cross-terms in this integral, the right-hand side is maximized at a degenerate distribution which maximizes $E_s[\pi_\delta(Q^R(x,s); s)]$. But this is exactly the problem of choosing optimal feasible deterministic supply given the reserve price $R$. It follows that expected revenue is weakly dominated by expected revenue with optimal deterministic supply, hence optimal supply is deterministic.

Remark 3. The proof of Theorem 5 remains valid for the profit maximization problem of a
seller facing increasing marginal costs. Let \( C(Q) \) be the seller’s cost of supplying quantity \( Q \), and assume that \( c(Q) = dC(Q)/dQ \) is positive and weakly increasing. Equation (11) for expected profits in the proof of Theorem 5 must be adjusted to

\[
\mathbb{E} \left[ \pi^F \right] = \mathbb{E}_s \int_0^Q \int_0^{Q^R(y,s)} p(x; R, s) - c(x) \, dx \, dF(y).
\]

Subsequent integration by parts remains valid, and equation (13) becomes

\[
\mathbb{E} \left[ \pi^F \right] = \mathbb{E}_s \int_0^{Q^R(s)} (1 - F(y)) (\hat{v}(y; s) - c(y)) + (1 - F(y)) \frac{1}{\pi} \int_y^{Q^R(s)} \hat{v}_q(x; s) (1 - F(x))^{n-1} \, dx \, dy.
\]

As before, letting \( \pi^\delta(q; s, c) \) be monopoly profits when quantity \( q \) is sold to type \( s \) given marginal cost curve \( c \), we obtain

\[
\mathbb{E} \left[ \pi^F \right] \leq \mathbb{E}_s \left[ \int_0^Q \pi^\delta \left( Q^R(x, s); s, c \right) \, dF(x) \right].
\]

The remainder of the proof is immediate.

E  Robust Selection and the Proofs for Section 5 (The Auction Design Game)

E.1 Robust and Semi-truthful Equilibria in Uniform Price

In the uniform-price auction, equilibrium bidding strategies are unique when the support of supply is sufficiently large as established by Klemperer and Meyer [1989]; for their argument to apply in our setting, it is sufficient that the support of supply contains \([0, \bar{Q}]\), where \( \bar{Q} \geq \sup_s n\nu^{-1}(R; s) \). Because the bids in Klemperer and Meyer’s equilibrium remain best responses even after the bidders learn the realization of supply, these bids remain in equilibrium for all supply distributions (assuming the reserve price is kept the same). This observation allows us to re-interpret Klemperer and Meyer’s uniqueness result as a selection of an equilibrium that is robust to bidders’ beliefs about the distribution of supply.

In the uniform-price auction, bids for quantities which are never marginal never affect utility, and are relevant only in ensuring that there is no profitable deviation from a particular best response bid curve. For example, when supply is deterministic bidders can coordinate on collusive-seeming equilibria, in which the market-clearing price is low, and high bids for nonmarginal units ensure it is not profitable for any bidder to increase their allocation by increasing their bid. The seller has the ability to nearly-costlessly eliminate these equilibria
by adding a small amount of randomness to aggregate supply, ensuring that all quantities remain potentially marginal. Robust bids are therefore focal in our equilibrium analysis of the uniform-price design game: bidders cannot credibly commit to bidding below robust bids, because the seller can introduce a small amount of randomness to induce (at worst) a robust bidding equilibrium.

**Lemma 13. [Symmetric Equilibrium in Uniform Price]** For all signals $s$ and any price $p(s) \in [R, v(Q^nR(s)/n; s)]$, there is a symmetric equilibrium of the uniform-price auction in which all bidders bid

$$b(q; s) = v(q; s) + \int_q^{1/Q^nR(s)} \left(\frac{q}{x}\right)^{n-1} v_q(x) \, dx - \left(\frac{q}{1/nQ^nR(s)}\right)^{n-1} \left(v\left(\frac{1}{nQ^nR(s)}; s\right) - p(s)\right).$$

**Proof.** We follow the approach of Klemperer and Meyer [1989]: they show that there is continuum of asymmetric equilibria in uniform price, and we leverage their analysis to show that all prices given above can be supported in symmetric equilibria. The first-order conditions of the uniform-price auction are

$$(v(Q - (n - 1)\varphi(p); s) - p) + \left(\frac{Q - (n - 1)\varphi(p)}{n - 1}\right)\frac{1}{\varphi_p(p)} = 0, \forall Q. \quad (15)$$

At the symmetric solution, $\varphi(p) = Q/n$. To show that $b(\cdot; s)$ is an equilibrium bidding function, it is sufficient to show that the bid function is feasible, and that the above expression is negative for $p' > p$ and positive for $p' < p$; equivalently, since the equality is solved at $\varphi(p) = Q/n$, for the latter point to hold it is sufficient to show that the above expression is negative for $Q > n\varphi(p')$ and positive for $Q < n\varphi(p')$. We then check

$$\text{sign} \left[ (v(Q - (n - 1)\varphi(p')) - p') + \frac{Q - (n - 1)\varphi(p')}{(n - 1)\varphi_p(p')} \right]$$

$$= \text{sign} \left[ (v(Q - (n - 1)\varphi(p')) - p') + \frac{Q - (n - 1)\varphi(p')}{(n - 1)\varphi_p(p')} - \left(v(n\varphi(p') - (n - 1)\varphi(p')) - p' + \frac{n\varphi(p')}{(n - 1)\varphi_p(p')}\right)\right] = 0$$

$$= \text{sign} \left[ (v(Q - (n - 1)\varphi(p')) - v(n\varphi(p') - (n - 1)\varphi(p'))) + \frac{Q - n\varphi(p')}{(n - 1)\varphi_p(p')} \right].$$

Recalling that $\varphi_p < 0$, when $Q < n\varphi(p')$ the leading and trailing expressions are positive, and when $Q > n\varphi(p')$ the leading and trailing expressions are negative, as desired.

The bid function is feasible if bids are everywhere above the reserve price and below marginal values. Any solution to the first-order condition such that the bid for the max-
imum quantity is within \([R, v(Q^R(s)/n; s)]\) is decreasing (hence everywhere above \(R\)) and bounded above by marginal values, and is therefore feasible. Woodward [2021] shows that the symmetric solution to 15 is

\[
b(q; s) = v(q; s) + \int_q^{\frac{1}{n}Q^R(s)} \exp \left( -\frac{(n-1)}{z} \int_q^x \frac{d\nu}{z} \right) v_q(x) \, dx - C \exp \left( -\frac{(n-1)}{\frac{n}{n}Q^R(s)} \int_q^x \frac{d\nu}{x} \right)
\]

\[
= v(q; s) + \int_q^{\frac{1}{n}Q^R(s)} \left( \frac{q}{x} \right)^{n-1} v_q(x) \, dx - \left( \frac{q}{\frac{n}{n}Q^R(s)} \right)^{n-1} C.
\]

Setting \(C\) so that \(b(Q^R(s)/n; s) = p(s)\) completes the proof.

The existence of semi-truthful and robust equilibria is an immediate consequence of Lemma 13. Proposition 1 gives the explicit form of robust equilibrium bids.

**Proposition 1. [Bids Robust to Uncertainty]** The unique uniform-price equilibrium bid profile that is robust to uncertainty is given by:

\[
b(q; s) = \left( \frac{q}{v^{-1}(R; s)} \right)^{n-1} R + (n-1) \int_q^{v^{-1}(R, s)} \left( \frac{q}{x} \right)^{n-1} \frac{v(x; s)}{x} \, dx,
\]

or, equivalently,

\[
b(q; s) = v(q) + \int_q^{v^{-1}(R, s)} \left( \frac{q}{x} \right)^{n-1} v_q(x) \, dx.
\]

**Proof.** With unbounded supply, expression (16) gives the unique solution to the equilibrium first-order conditions in the uniform-price auction (Lemma 13). Then \((b^i)_{i=1}^n\) is the unique robust uniform-price bid profile.

We henceforth refer to the above uniform-price bid function as the robust uniform-price bid. The robust uniform-price bid is continuous, differentiable, strictly below marginal values for all \(q \in (0, v^{-1}(R; s))\), and equal to marginal values for \(q \in \{0, v^{-1}(R; s)\}\). No matter which auction format is employed, optimal supply \(Q^* > 0\). In the pay-as-bid design game the optimal deterministic quantity must be binding for some bidder types, \(Q^{PAB} < \sup_s v^{-1}(R; s)\), provided the value space is rich. Since robust uniform-price bids are strictly below value on \((0, Q^{PAB}/n]\) for all types \(s\) such that \(Q^{PAB} < v^{-1}(R; s)\), the pay-as-bid auction generates strictly greater revenue than the uniform-price auction with robust bidding. Because, in the auction design game, bidders can select an equilibrium on the basis of the supply and reserve chosen by the auctioneer, revenue dominance of deterministic pay-as-bid is sufficient to prove Lemma 1.
Proof of Lemma 1. We first show that, holding bids fixed, optimal supply is deterministic in the uniform-price auction. Given bid $b$ and distribution of per-capita supply $F^\mu$, the expected revenue obtained from a given bidder in the uniform-price auction is

$$
E_s \left[ \left(1 - F^\mu \left(\frac{Q^R}{s} \right) \right) RQ^R \left(\frac{Q^R}{s} \right) + \int_0^{\frac{Q^R}{s}} qb \left(q; s \right) dF^\mu \left(q \right) \right]
$$

$$
= E_s \left[ \int_0^{\frac{Q^R}{s}} b \left(q; s \right) + qb \left(q; s \right) \left(1 - F^\mu \left(q \right) \right) dq \right]
$$

$$
= \int_0^{\frac{Q^R}{s}} E_s \left[ b \left(q; s \right) + qb \left(q; s \right) \left| Q^R \left(\frac{Q^R}{s} \right) > q \right\} \left(1 - F^\mu \left(q \right) \right) dq = \int_0^{\frac{Q^R}{s}} J \left(q; s \right) \left(1 - F^\mu \left(q \right) \right) dq.
$$

It follows that the optimal distribution $F^\mu$ is deterministic, and is equal to 0 below some threshold and 1 above it.

Following Proposition 1, robust uniform-price bids can be represented as

$$
b \left(q; s \right) = v \left(q; s \right) + \int_q^{\frac{Q^R}{s}} \frac{1}{x} v_q \left(x \right) dx.
$$

Because $v_q < 0$, these bids are strictly below values at all $q < \frac{Q^R}{s}$. And because optimal supply (holding fixed bids) is deterministic, optimal revenue under robust bids is strictly below optimal pay-as-bid revenue: otherwise there is a reserve $R$ and deterministic quantity $Q$ that yield expected uniform-price revenue equal to expected pay-as-bid revenue, contradicting the richness of the value space.

Since the maximum expected revenue obtained under robust bids is strictly below the optimal expected revenue in the pay-as-bid auction, it is sufficient to show that when $\varepsilon > 0$ is small and random supply is supported on $[Q^{\text{PAB}} - \varepsilon, Q^{\text{PAB}} + \varepsilon]$ and the reserve price is $R \in [R^{\text{PAB}} - \varepsilon, R^{\text{PAB}} + \varepsilon]$, equilibrium uniform-price revenue under semi-truthful bids is close to optimal pay-as-bid revenue. A lower bound on this revenue is

$$
E_s \left[ \int_{Q^{\text{PAB}} - \varepsilon}^{Q^{\text{PAB}} + \varepsilon} Q^R \left(Q; s \right) v \left(\frac{1}{n} Q^R \left(Q; s \right) ; s \right) dF \left(Q \right) \right] \geq E_s \left[ Q^R \left(Q^{\text{PAB}} - \varepsilon; s \right) v \left(\frac{1}{n} Q^R \left(Q^{\text{PAB}} - \varepsilon; s \right) ; s \right) \right].
$$

Because $Q^R$ is continuous in $Q$ and $R$, and because $R \to R^{\text{PAB}}$ as $\varepsilon \searrow 0$, the right-hand side converges to

$$
E_s \left[ Q^{R^{\text{PAB}}} \left(Q^{\text{PAB}}; s \right) v \left(\frac{1}{n} Q^{R^{\text{PAB}}} \left(Q^{\text{PAB}}; s \right) ; s \right) \right].
$$

This is exactly optimal pay-as-bid revenue. Then suppose that bidders play semi-truthful bids when the auctioneer selects reserve $R$ and distribution $F$, and play robust bids otherwise.
Provided $\varepsilon > 0$ is sufficiently small, reserve $R$ and distribution $F$ will yield more revenue to the auctioneer than any other selection. The result follows.

Proposition 2. [Strict Dominance of Pay-as-Bid Revenue] The pay-as-bid design game generates strictly greater revenue than the unique robust equilibrium of the uniform-price design game.

Proposition. [Range of Prices in Uniform Price] If $p^*(s)$ is the market-clearing price at supply $Q_R$ in an equilibrium of the uniform-price auction with supply distribution $F$, then for all signals $s$, $p^*(s) \in [R, v(Q_R/n; s)]$. Furthermore, for any supply distribution, any signal $s$, and any $p^*(s) \in [R, v(Q_R/n; s)]$, there is an equilibrium of the uniform-price design game with market-clearing price at supply $Q_R$ equal to $p^*(s)$.

Proof. The second claim follows from Lemma 13. To prove the first claim note that $p^*(s) \geq R$ by definition. If $p^*(s) > v(Q_R/n; s) \geq R$, then with positive probability some bidder $i$ is allocated a positive mass of $q_i$ such that $v(q_i; s) < p^*(s)$. If this bidder bids $b' = v(\cdot; s)$ instead, she is awarded all units she values above $p^*(s)$ at a price no greater than $p^*(s)$. This deviation is profitable as she keeps all positive-margin units, drops the negative-margin units, and does not increase her payment.

E.2 Deterministic Revenue Bound in Uniform Price

Lemma 14. [Deterministic Dominance in Uniform Price] For any equilibrium of the uniform-price design game $((R, F), b)$, there is a deterministic-supply equilibrium $((R^*, F^*), b^*(\cdot; s, R^*, F^*))$ that generates weakly higher seller revenue and has the same on-path bids.

Proof. With symmetrically-informed bidders, equilibrium bids in the uniform-price auction are optimal for every realization of supply, a point first made by Klemperer and Meyer [1989]. For a given bidder, every realization of supply determines a residual supply curve corresponding to the demands of the other bidders, and the given bidder’s bid effectively serves to select the price-quantity pair from this residual supply curve; this choice does not depend on choices at other realizations of supply as long as the resulting bid curve is downward-sloping. In effect, two supply distributions with the same support admit the same set of equilibria, and if one supply distribution has a smaller support than another, its set of equilibrium bids is a weak superset of the other. This implies that the revenue-maximizing equilibrium with deterministic supply is also revenue-maximizing among all possible equilibria.
E.3 Proof of Theorem 7

In the proof below we decorate market outcome functions with superscripts denoting the relevant mechanism, where helpful. For example, \( p^{\text{UP}} \) is the market-clearing price in the uniform-price auction and \( p^{*\text{PAB}} \) is the market-clearing price in the pay-as-bid auction.

**Proof of Theorem 7.** As discussed in Theorem 5 and Lemma 14, we may restrict attention to optimal deterministic supply distributions in both the pay-as-bid and uniform-price auctions. Revenue maximization may then be expressed as a per-agent quantity \( q^* \) and market price \( p^* \); for signals \( s \) such that \( v(q^*; s) \geq p^* \) it is without loss to assume that the total allocation is \( nq^* \)—there is sufficient demand for the total quantity at the reserve price—while for signals \( s \) such that \( v(q^*; s) < p^* \) it is clear that some total quantity \( nq' < nq^* \) will be allocated. The seller’s expected revenue is then an expectation over bidder signals,

\[
\mathbb{E}_s [\pi] = \mathbb{E}_s [nq(q^*, p^*; s) \cdot p(q^*, p^*; s)].
\]

The quantity allocated under the uniform-price auction equals the quantity allocated under the pay-as-bid auction, \( q^{\text{UP}}(q^*, p^*; s) = q^{\text{PAB}}(q^*, p^*; s) \), whenever \( v(\cdot; s) \) is strictly decreasing at this quantity, or when \( v(\cdot; s) > p^* \) at this quantity. Since we have assumed that \( v(\cdot; s) \) is strictly decreasing, the quantity allocation depends only on \( q^* \) and \( p^* \) and not on the mechanism employed. Additionally, it is the case that \( p^{*\text{UP}}(q^*, p^*; s) = p^{*\text{PAB}}(q^*, p^*; s) \) whenever \( v(q^*; s) < p^* \). Let \( \mathcal{S} \) be the set of such \( s \),

\[
\mathcal{S} = \mathcal{S}(nq^*, p^*) = \{s' : v(q^*; s) < p^* \}.
\]

Then we have

\[
\mathbb{E}_s [\pi] = p^* \Pr (s \in \mathcal{S}) \mathbb{E}_s [nq(q^*, p^*; s)|s \in \mathcal{S}] + nq^* \Pr (s \notin \mathcal{S}) \mathbb{E}_s [p(q^*, p^*; s)|s \notin \mathcal{S}].
\]

The left-hand term is independent of the mechanism employed, while the right-hand term depends on the mechanism only via the expected market-clearing price. In the pay-as-bid auction, we have seen that \( p(q^*, p^*; s) = v(q^*; s) \) for all \( s \notin \mathcal{S} \), while in the uniform-price auction any price \( p \in [p^*, v(q^*; s)] \) is supportable in equilibrium. It follows that the pay-as-bid auction weakly revenue dominates the uniform-price auction, and generally will strictly do so. That the seller-optimal equilibrium of the uniform-price auction is revenue-

\[87\]

\[89\]In the latter case there is excess demand, so all units will be allocated. In the former case all units are allocated at the reserve price; there is a possible difference in allocation when bidders’ marginal values are flat over an interval of quantities at the reserve price, since bidders are indifferent between receiving and not receiving these quantities.
equivalent to the unique equilibrium of the pay-as-bid auction arises from the selection of 
\( p^{\text{UP}}(q^*, p^*; s) = v(q^*; s) \) for all \( s \notin \mathcal{S} \).

F Proof of Large Market Revenue Equivalence

Proposition 3. [Large Market Revenue Equivalence] If per-capita aggregate supply \( F^\mu \) has full support, equilibrium expected revenue in the pay-as-bid and uniform-price auctions converges to the expected revenue in a uniform-price auction with truthful reporting as the number of bidders grows large.

Proof. Equilibrium bids in the pay-as-bid auction, given in Theorem 3, converge as \( n \to \infty \); the same is true of equilibrium bids in the uniform-price auction, given in Lemma 13. Moreover, equilibrium bids in the uniform-price auction with full support converge to truthful reporting.

It remains to be seen that equilibrium per capita expected revenue in the pay-as-bid auction with many bidders is equal to per capita expected revenue in the uniform-price auction with truthful reporting. Letting \( q(Q; R, s) = \min\{Q, v^{-1}(R; s)\} \), we establish the claim pointwise for each signal \( s \),

\[
\int_0^{\overline{Q}} q(Q; R, s) v(q(Q; R, s); s) dF^\mu(Q) = \int_0^{\overline{Q}} \int_0^{Q} v(q(x; R, s); s) dF^{Q', \mu}(x) dQ' dF^\mu(Q).
\]

Integration by parts of the innermost integral rearranges the right-hand side to

\[
\int_0^{\overline{Q}} \int_0^{Q} v(q(Q; R, s); s) + \int_{\overline{Q}} v(q(x; R, s); s) \frac{1 - F^\mu(x)}{1 - F^\mu(Q')} dQ' dF^\mu(Q).
\]

Subsequent integration by parts of the second integral rearranges the right-hand side to

\[
\int_0^{\overline{Q}} Qv(q(Q; R, s); s) + Q \int_0^{\overline{Q}} \left[ x < v^{-1}(R; s) \right] v(q(x; R, s); s) \frac{1 - F^\mu(x)}{1 - F^\mu(Q)} dx
\]

\[
- \int_0^{Q} \int_{Q'} \left[ x < v^{-1}(R; s) \right] v(q(x; R, s); s) \frac{1 - F^\mu(x)}{(1 - F^\mu(Q'))^2} dxdQ' dF^\mu(Q).
\]

Because \( \int_0^{\overline{Q}} Qv(q(Q; R, s); s) dF^\mu(Q) \) is expected revenue in the uniform-price auction with
truthful reporting, it is sufficient to show that

$$
\int_0^{\hat{Q}'} Q \int_Q^{\hat{Q}'} 1 \left[ x < v^{-1} (R; s) \right] v_q (q (x; R, s); s) \frac{1 - F^\mu (x)}{1 - F^\mu (Q)} dxdF^\mu (Q)
= \int_0^{\hat{Q}'} \int_0^Q Q' \int_{Q'}^{\hat{Q}'} 1 \left[ x < v^{-1} (R; s) \right] v_q (q (x; R, s); s) \frac{(1 - F^\mu (x)) f^\mu (Q')}{(1 - F^\mu (Q'))^2} dxdQ'dF^\mu (Q).
$$

To streamline exposition, define

$$
J (Q) = \int_Q^{\hat{Q}'} 1 \left[ x < v^{-1} (R; s) \right] v_q (q (x; R, s); s) \frac{1 - F^\mu (x)}{1 - F^\mu (Q)} dx.
$$

Then the desired equality is

$$
\int_0^{\hat{Q}'} QJ (Q) dF^\mu (Q) = \int_0^{\hat{Q}'} \int_0^Q Q' J (Q') \frac{f^\mu (Q')}{1 - F^\mu (Q')} dQ'dF^\mu (Q).
$$

Integration by parts of the outer integral rearranges the right-hand side to

$$
\int_0^{\hat{Q}'} QJ (Q) f^\mu (Q) dQ = \int_0^{\hat{Q}'} QJ (Q) dF^\mu (Q).
$$

Then per capita expected revenue in a large pay-as-bid auction is identical to per capita expected revenue in a large uniform-price auction with truthful reporting.

\[ \square \]

## G Proofs for Appendix A (Elastic Supply)

### G.1 Proof of Theorem 9 (Uniqueness with Elastic Supply)

**Proof.** The analysis from the proof of Theorem 1 allows us to conclude that on the maximum unit each bidder might receive, the bidder pays her marginal value. Letting \( \hat{Q} (s) \) be the aggregate quantity awarded in equilibrium under supply curve \( Q^*(s) \), it cannot be that \( p^* (\hat{Q} (s); s) > \hat{v} (\hat{Q} (s); s) \), since bids are below values. If, instead, \( p^* (\hat{Q} (s); s) < \hat{v} (\hat{Q} (s); s) \), the arguments from the proof of Theorem 1 apply; indeed, they are strengthened by the fact that a small increase in bid increases allocation not only by beating opponent bids, but also by increasing the market price and moving up the supply curve.

Because each bidder bids \( b^* (\hat{Q} (s)/n; s) = v (\hat{Q} (s)/n; s) \) in any equilibrium, each bidder’s allocation is \( \hat{Q} (s)/n \). This allocation is deterministic, conditional on the bidder-common signal \( s \). Then any bid curve \( b \) such that \( b (q) > v (\hat{Q} (s)/n; s) \) for some \( q > 0 \) is wasteful: it does not affect the resulting allocation, and \( \int_0^{\hat{Q}(s)/n} b (q) dq > \int_0^{\hat{Q}(s)/n} b^* (q; s) dq \). It follows
that \( b^*(q) = v(\hat{Q}(s)/n; s) \) for all \( q \leq \hat{Q}(s)/n \), and equilibrium bids are unique for all relevant quantities.

\[ \square \]

### G.2 Proof of Lemma 2

As we consider the special case of the seller who knows the bidders’ values, we simplify notation and suppress the signal while writing value and bid functions.

#### G.2.1 Preliminary Definitions

Recall that we defined the supply reserve distribution \( K(Q; R) \) in Appendix A. For simplicity, we carry out the analysis under the assumption that supply-reserve distribution \( K \) is continuously differentiable. We show that this assumption may be dropped in Remark 4.

Holding the supply-reserve distribution \( K \) fixed, fix a bidder \( i \) and consider the aggregate demand of her opponents. Allowing for mixed strategies and asymmetric and asymmetrically-informed bidders, we denote the aggregate demand of bidder \( i \)'s opponents by \( Q(\cdot; \xi) \), where \( \xi \) indexes the joint distribution of her opponents’ potentially mixed strategies. As with supply-reserve distribution \( K \), we assume that aggregate demand \( Q \) is continuously differentiable, and drop this assumption in Remark 4. Although we separately specify the supply-reserve distribution \( K \) and the mixed strategy index \( \xi \) because the former is controlled by the seller while the latter is not, the set of bidder’s best responses does not depend on the source of randomness in a bidder’s residual supply. Bidders’ ex post utility is determined by realized quantity and payment, and thus the dependence of interim utility on the joint distribution of quantity and payment is unaffected by the introduction of a random reserve price, asymmetry and asymmetric information among bidders, and the possibility of mixed strategies. Thus, the bidder’s first order condition is unchanged from the analysis in Lemma 9 (in Supplementary Appendix B), and random reserve affects only the distribution of realized quantity. In the language of Lemma 9,

\[
G^i(q; b) = \mathbb{E}_\xi [K(q + Q(b; \xi); b)],
\]

and

\[
G^i_b(q; b) = \mathbb{E}_\xi [K_Q(q + Q(b; \xi); b) Q_p(b; \xi) + K_R(q + Q(b; \xi); b)].
\]

(17)

E.g. when the reserve price is fixed, \( K_R = 0 \) for all relevant prices, and (17) is identical to what we find in equation (8).

We aim to show that the seller can induce the same bidder behavior by implementing a random reserve without constraining supply, in which case \( K_Q = 0 \), and the bidder’s
pointwise first order condition is
\[(v(q) - b(q)) \mathbb{E}_\xi [K_R (q + Q(b(q); \xi); b)] = \mathbb{E}_\xi [K(q + Q(b(q); \xi); b)].\]

As \(K_Q = 0\) implies that \(K\) is independent of \(q\) (and thus \(Q\) is independent of \(\xi\)), we write this in terms of only the distribution of reserve prices \(F^R\),
\[(v(\varphi(p)) - p) F^R_p (p) = F^R (p).\]

Thus a key simplification associated with random reserve and unconstrained supply is that the optimal bid is determined by the reserve distribution \(F^R\) and does not depend on opponent bids. Furthermore, for each quantity the optimal bid is either pointwise optimal, or this quantity is part of an interval on which the first order conditions are ironed, cf. Woodward [2016]. We capture these optimality conditions in the concept of first-order optimal bids defined as follow.

**Definition 2.** Given a distribution of reserve prices \(F^R\), we say that \(b\) is first-order optimal with respect to \(F^R\) if:

1. If \(b\) is strictly decreasing at \(q\), then it solves the pointwise first order condition: \((v(q) - b(q)) F^R_p (b(q)) = F^R (b(q)).\)

2. If \(b\) is constant in a neighborhood of \(q\) then \(b(q)\) is a mass point of \(F^R\) and it solves the ironed first order condition:
\[
\left( F^R (b(q)) - F^R (\bar{b}) \right) (v(\varphi(p)) - \bar{b}) = (b(q) - \bar{b}) F^R (p), \text{ where } \bar{b} = \lim_{q' \searrow \varphi(p)} b(q').
\]

Intuitively, the ironing conditions state that the marginal gain from slightly extending the constant interval (marginal additional quantity with probability \(F^R (b(q)) - F^R (\bar{b})\)) must equal the marginal cost from the same (marginal additional payment with probability \(F^R (\bar{b})\)).

As \(b\) is weakly decreasing, any quantity \(q\) belongs to either an interval on which \(b\) is flat or to an interval on which \(b\) is strictly decreasing (and it might be an endpoint of both types of intervals simultaneously). The structure of these intervals can be complex, but there is at most a countable number of them.

Although optimal bids are first-order optimal the converse need not be true: first-order optimality only implies that a bid satisfies pointwise first order conditions where applicable, and ironing conditions elsewhere. In deriving the revenue bounds below, we assume only that the first-order conditions are satisfied, not that bids are optimal. Because any (glob-
ally) optimal bid function satisfies the first-order optimality conditions above, the bound on revenues applies to optimal bids.

Let \( G^K(\cdot; b, Q) \) be the distribution of realized quantity given stochastic-elastic supply \( K \), bid function \( b \), and, potentially random, residual supply \( Q \), and let \( G^R(\cdot; b) \) be the distribution of realized quantity given reserve distribution \( F^R \) and bid function \( b \). As mentioned above, \( G^R \) does not depend on \( Q \) because, under random reserve, supply does not depend on opponent bids. Letting \( \xi \) represent randomness in residual supply (e.g., mixed strategies for a bidder’s opponents)\(^90\) we have

\[
G^R (q; b) = 1 - F^R (b (q)),
\]

\[
\frac{d}{dq} G^R (q; b) = -F^R_p (b (q)) b_q (q);
\]

\[
G^K (q; b, Q) = \mathbb{E}_\xi [K (q + Q (b (q) ; \xi) , b (q))],
\]

\[
\frac{d}{db} G^K (q; b, Q) = \mathbb{E}_\xi [K_q (q + Q (b (q) ; \xi)) Q_p (b (q) ; \xi) + K_p (q + Q (b (q) ; \xi) , b (q))],
\]

\[
\frac{d}{dq} G^K (q; b, Q) = \frac{d}{db} G^K (q; b, Q) b_q (q) + \mathbb{E}_\xi [K_q (q + Q (b (q) ; \xi))].
\]

The expected revenue from bidder \( i \) when the bidder bids \( b \) and the bid leads to quantity distribution \( G^* \) is given by \( \pi (b; G^*) = \int_0^Q \int_0^q b (x) dx dG^* (q) \).

### G.2.2 The Optimality of Random Reserve with Known Values

We begin with a bid function \( b \) which is a best response to residual supply distribution \( G^i(\cdot; b) \) and construct a reserve price distribution and bidder’s best response to this new distribution that raise more revenue.

**Lemma 15.** Let \( b \) be a best response bid curve under residual supply distribution \( G^i \), generated by supply-reserve distribution \( K \) and stochastic aggregate demand \( Q \). There is a reserve distribution \( F^R \) and a first order best response \( b^R \) to \( F^R \) such that \( \pi \left( b^R; G^R \right) \geq \pi \left( b; G^i \right) \).

While the bound on revenue in Lemma 15 might depend on the equilibrium selected, the subsequent analysis will show that this bound is weakly lower than the revenue in a unique equilibrium under deterministic elastic supply.

**Proof.** For clarity, we proceed under the assumption that supply-reserve distribution studied \( K \) and aggregate residual demand \( Q \) are continuously differentiable. Following the derivation

\(^90\)In the main text we focus on pure strategies. In this analysis we allow for mixed strategies, allowing us to show that all randomness—exogenous or otherwise—is detrimental to the seller’s revenue.
of the result for smooth $K$ and $Q$, we comment on extending the argument to potentially discontinuous $K$ and $Q$.

Any distribution of reserve $F^R$ and the first-order optimal response $b^R$ induce a distribution (c.d.f.) $\tilde{G}^R$ of quantities sold to bidder $i$. As a step in constructing $b^R$ and $F^R$, we first construct an auxiliary distribution $G^R$ which is not necessarily equal to $\tilde{G}^R$. As a preparatory step to construct the latter distribution, recall that the discussion of the previous subsection shows that under a random reserve price that induces differentiable quantity distribution $G^R$, in any interval in which $b$ is strictly decreasing. We will define $G^R$ so that the pointwise first order conditions of an agent bidding $b$ are satisfied; that is,

$$-(v(q) - b(q))G_q^R(q) = \left(1 - G^R(q)\right)b_q(q),$$

and thus

$$\frac{d}{dq} \ln \left[1 - G^R(q)\right] = \frac{b_q(q)}{v(q) - b(q)}.$$  

Given any initial value of $G^R(q)$ (initial condition of the ODE), we can solve this differential equation for any differentiable $b < v(q)$. In particular, for any quantity $\bar{q}$ such that $b$ is strictly decreasing on $(\bar{q}, q)$, we obtain

$$G^R(\bar{q}) = 1 - \exp\left(\int_{q}^{\bar{q}} \frac{b_q(x)}{v(x) - b(x)} dx\right) \left[1 - G^R(q)\right]. \tag{19}$$

We now construct $G^R$ and we show that $G^R \succeq_{FOSD} G^K$; in particular, $G^R$ puts more weight on larger quantities than $G^K$ does. To start, let $G^R(0) = G^K(0)$. At the left endpoint of any maximal interval $(\bar{q}_\ell, \bar{q}_r)$ on which $b$ is strictly decreasing, we define $G^R$ so that $G^R(\bar{q}_\ell) = G^K(\bar{q}_\ell)$, and we define $G^R$ on the interior of $(\bar{q}_\ell, \bar{q}_r)$ so that $b$ satisfies the first-order ODE given the initial condition $G^R(\bar{q}_\ell)$.\footnote{An interval is maximal with respect to a given property if there is no larger, inclusive interval that also satisfies the property.} In particular, the first-order ODE determines the value at the right endpoint of the strictly decreasing $b$ interval, $G^R(\bar{q}_r)$. For any maximal open interval $(q_\ell, q_r)$ on which $b$ is constant, let the value at the right endpoint be $G^R(q_r) = G^K(q_r)$.\footnote{Note that $G^R$ is well defined if it so happens that $q_r = \bar{q}_r$.} Notice that for any maximal interval $(q_\ell, q_r)$ on which $b$ is constant, $q_\ell$ is either 0 or equal to a limit of a sequence of the right end points of maximal intervals.\footnote{Notice that the limit might be over right end-points of both strictly decreasing $b$ and constant $b$ intervals. We of course allow for a constant sequence, that is the case where $q_\ell$ is the right end point of an adjacent interval.} We will see below that the values of $G^R$ on this sequence are monotonic. Since they are also bounded below (they are nonnegative), the sequence of values of $G^R$ at these right endpoints converges, and
we define $G^R(q_t)$ as its limit, and also set $G^R(q) = G^R(q_t)$ for $q$ in in the interior of any maximal open interval $(q_t, q_r)$ on which $b$ is constant. This concludes the construction of $G^R$ for all quantities strictly lower than the maximum possible quantity; at this quantity we set $G^R$ to equal 1. Thus $G^R$ is a c.d.f. iff it is monotonic.

To establish monotonicity, suppose that $q_t$, $q_r$ are such that $q_t < q_r$, $G^R(q_t) \leq G^K(q_t)$, and that $b$ is strictly decreasing on $(q_t, q_r)$. Then on $(q_t, q_r)$, the pointwise first-order optimality conditions obtain, and we have

$$-(v(q) - b(q))G^R_q(q) = \left(1 - G^R(q)\right)b_q(q), \quad \text{and} \quad -(v(q) - b(q))G^K_b(q) = 1 - G^K(q);$$

in particular, $G^R$ and $G^K$ are continuous on $(q_t, q_r)$. The left-hand equation holds by construction of $G^R$ and the right-hand equation follows from the fact that $b$ is a best response to supply-reserve distribution $K$ and opponent demand $Q$. By construction, the $-(v(q) - b(q))$ terms are equal, and so for any $q \in (q_t, q_r)$ it must be that

$$\frac{G^R_q(q)}{1 - G^R(q)} = \frac{G^K_b(q)b_q(q)}{1 - G^K(q)}. \quad (20)$$

Suppose that there is $q \in (q_t, q_r)$ such that $G^R(q) > G^K(q)$. Then there is $\hat{q} \in (q_t, q)$ such that $G^R(\hat{q}) = G^K(\hat{q})$, because the c.d.fs $G^R$ and $G^K$ are continuous on $(q_t, q_r)$ and $G^R(q_t) \leq G^K(q_t)$. At this $\hat{q}$, equation 20 becomes $G^R_q(\hat{q}) = G^K_b(\hat{q})b_q(\hat{q})$, and substituting in for equations 18 gives

$$G^R_q(\hat{q}) = G^K_b(\hat{q})b_q(\hat{q}) = G^K_q(\hat{q}) - \mathbb{E}_\xi[K_q(q + Q(b(q); \xi))] \leq G^K_q(\hat{q}).$$

We conclude that $G^K(\hat{q}) = G^R(\hat{q})$ implies $G^K_q(\hat{q}) > G^R_q(\hat{q})$, contradicting $G^R(q) > G^K(q)$. From this it follows that if $b$ is strictly decreasing on $[q_t, q_r]$ and $G^R(q_t) \leq G^K(q_t)$, then $G^R|_{q \in [q_t, q_r]} \geq_{\text{FOSD}} G^K|_{q \in [q_t, q_r]}$, and, in particular, $G^R(q_r) \leq G^K(q_r)$. This inequality allows us to conclude that if $q_r$ is the limit of left endpoints $\hat{q}_t > q_r$ of maximal intervals, then $G^R(q_r)$ is weakly below the limit of $G(\hat{q}_t)$ on this sequence. We can conclude that that $G^R$ is monotonic and hence a cumulative distribution function such that $G^R \geq_{\text{FOSD}} G^K$.

We now define the random reserve distribution $F^R$ as follows: for any $q$, let $F^R(b(q)) = 1 - G^R(q)$. We construct $b^R$ that is first-order optimal bid function with respect to $F^R$ and such that $b^R \geq b$. Our construction is iterative: we begin with $b^{R_0} = b$, then show how to compute $b^{R_t+1}$ from $b^{R_t}$ for any $t \geq 0$. Let $\Omega_t$ be the set of maximal constant intervals of $b^{R_t}$. For an interval $(q_t, q_r) \in \Omega_t$, let $\hat{q}_r$ solve the ironed first-order optimality condition at
bid level $b^{Rt}(q_r)$:

$$
(F^R (b^{Rt}(q_r)) - \lim_{q \searrow q_r} F^R (b^{Rt}(q))) (v(q_r) - b^{Rt}(q_r)) = \left( b^{Rt}(q_r) - b^{Rt}(\tilde{q}_r) \right) F^R \left( b^{Rt}(\tilde{q}_r) \right).
$$

Since $p = b^{Rt}(q_r)$ is a level at which $b$ is constant, there is a mass point in $F^R$ at $b^{Rt}(q_r)$, and the first-order ironing equation cannot be solved at $\tilde{q}_r < q_r$. It follows that $\tilde{q}_r \geq q_r$, and moreover that $b^{Rt}(\tilde{q}_r) \leq v(\tilde{q}_r)$. Then let $\hat{\Omega}_t$ be the set of intervals $(q_t, \tilde{q}_r)$, where $(q_t, q_r) \in \Omega_t$ and $\tilde{q}_r$ is derived from $q_r$ as above. We now define $b^{R[t+1]}$,

$$
\begin{align*}
   b^{R[t+1]}(q) &= \begin{cases} 
   \sup \{ b^{Rt}(q_r) : q \in (q_t, \tilde{q}_r) \in \hat{\Omega}_t \} & \text{if } \exists (q_t, \tilde{q}_r) \in \hat{\Omega}_t \text{ with } q \in (q_t, \tilde{q}_r), \\
   b^{Rt}(q) & \text{otherwise.}
\end{cases}
\end{align*}
$$

By construction, $b^{Rt} \leq b^{R[t+1]} \leq v$, and thus $b^{Rt} \to b^R$ for some $b^R$. Where the limit $b^R$ is strictly decreasing, it is equal to $b$ and therefore satisfies the first-order conditions for optimality. When the limit $b^R$ is constant, it satisfies the iterated first-order conditions for optimality by construction. It follows that $b^R$ is first-order optimal. Finally, since $b = b^{R0}$ and $b^{Rt} \leq b^{R[t+1]}$ for all $t$, it must be that $b \leq b^R$.

Being weakly higher than $b$, the bid function $b^R$ induces a realized quantity distribution $\tilde{G}^R$ that is weakly stronger than $G^R$ (the distribution of realized quantity with reserve distribution $F^R$ and bid $b$), which is in turn weakly stronger than $G^K$, and it follows that

$$
\pi(b^R; \tilde{G}^R) \geq \pi(b; G^K).
$$

Since $F^R$ implements $b^R$ as a first-order optimal bid function, the lemma follows.

\begin{remark}
When supply-reserve distribution $K$ and aggregate supply $Q$ are discontinuous, we adjust the first condition of the definition of a bidder’s first-order optimality at points at which $G^K$ is not differentiable and require at these points that the the left derivative in $b$ (which always exists, since $G^K$ is decreasing in $b$) satisfies

$$
-(v(q) - b(q)) G^i_{b-}(q; b) - \left(1 - G^K(q; b)\right) \geq 0.
$$

This is the only adjustment in the definition; the old definition is unchanged at points of
\end{remark}

\footnote{Measure-zero changes in bid do not affect utility. Therefore in this proof we assume, without loss of generality, that $b^{Rt}$ is left continuous.}

\footnote{Note that in the simple case where the original bid function $b$ is strictly decreasing, it is the case that $b^R = b$. The iterative process applied here handles the possible need to extend to the right the constant intervals from the original bid function $b$, as well as the possibility that one constant interval “overtakes” another in the iterative process. Note that in the latter case $b^R(q) > b(q)$ for $q$ in the overtaken constant interval of $b$.}

\footnote{The left derivative of a function $h$ at $x$ is defined as $h_{x-}(x) = \lim_{\varepsilon \searrow 0} (h(x) - h(x - \varepsilon))/\varepsilon$. Similarly the right derivative equals $h_{x+}(x) = \lim_{\varepsilon \nearrow 0} (h(x + \varepsilon) - h(x))/\varepsilon$.}
differentiability and where bids are flat. We follow the construction of $G^R$ in the proof of Lemma 15 with two adjustments: (i) we substitute the left derivative $G^i_{b_-}$ for derivative $G^i_b$, and (ii) the differential part of the construction is separately conducted for maximal intervals $(q_\ell, q_r)$ on which $b$ is strictly decreasing and continuous (as opposed to merely strictly decreasing). In this way, we are able to construct $G^R$ for all relevant quantity and price pairs, subject to verifying monotonicity like in the above proof of Lemma 15.

The monotonicity continues to hold because $G^K$ is monotonic and whenever $b$ is strictly decreasing and continuous, we have

$$0 = - (v(q) - b(q)) \frac{G^R_q(q)}{b_{q+}(q)} - \left(1 - G^R(q)\right) \leq - (v(q) - b(q)) G^K_{b_-}(q; b, Q) - \left(1 - G^K(q; b, Q)\right).$$

(21)

For any maximal interval $(q_\ell, q_r)$ on which $b$ is continuous and strictly decreasing we prove monotonicity by contradiction, as before. If there is $q \in (q_\ell, q_r)$ such that $G^R(q) > G^K(q)$, there is $\hat{q} \in [q_\ell, q_r]$ such that $G^R(\hat{q}) = G^K(\hat{q})$: even though $G^K$ is potentially discontinuous, $G^R$ is guaranteed to be continuous on the maximal interval in question (it is the solution to a differential equation) and $G^K$ is monotone. At this $\hat{q}$, plugging equations 18 into inequality 21 gives

$$G^K_{b_-}(\hat{q}) \leq \frac{G^R_q(\hat{q})}{b_{q+}(\hat{q})}.$$

Since $b$ is decreasing in $q$, this gives

$$G^R_q(\hat{q}) \leq G^K_{b_-}(\hat{q}) b_{q+}(\hat{q}) = G^K_{q+}(\hat{q}) - \mathbb{E}_\xi [K_{q+} (q + Q (b(q); \xi))] \leq G^K_{q+}(\hat{q}).$$

The final inequality follows from the fact that the exogenous supply-reserve distribution $K$ satisfies $K_{q+} \geq 0$. Then $dG^R(q; b, Q)/dq \leq dG^K(q; b, Q)/dq$ at $q = \hat{q}$, contradicting $G^R(q) > G^K(q)$ for some $q > \hat{q}$. The remainder of the proof follows the same steps as the original proof of Lemma 15.

### G.2.3 Approximation by Strictly-Decreasing Bid Functions

We now show that we can arbitrarily approximate the first-order optimal bid $b^R$ associated with random reserve $F^R$ with a strictly decreasing bid function $\tilde{b}^R$, associated with some random reserve distribution $\tilde{F}^R$, and that the distribution of realized quantity under this approximation approximates the distribution of quantity under $b^R$. Then since $b^R \geq b$ and $\tilde{b}^R \approx b^R$, it follows that $\tilde{b}^R$ approximates the revenue generated by $b$ under reserve distribution $F^R$ arbitrarily closely, or yields higher revenue.
Lemma 16. Given a reserve distribution $F^R$ with first-order optimal bid $b^R$ and any $\lambda > 0$, there is a reserve distribution $\tilde{F}^R$ with a strictly decreasing first-order optimal bid $\tilde{b}^R$ such that $\pi(\tilde{b}^R; G^R) > \pi(b^R, G^R) - \lambda$.

Proof. If $b^R$ is strictly decreasing the claim is trivially satisfied. Therefore, assume that $b^R$ is constant on the (maximal) interval $(q_\ell, q_r)$. Let $\tilde{b}^R \leq b^R$ be strictly decreasing on $(q_\ell, q_r)$ and such that $\tilde{b}^R_{|q_\notin(q_\ell, q_r)} = b^R_{|q_\notin(q_\ell, q_r)}$ and $\tilde{b}^R(q_r) = \lim_{q \nearrow q_r} b^R(q')$. Let $\tilde{F}^R|_{p > \tilde{b}^R(q_r)} = F^R|_{p > b^R(q_r)}$. Then $\tilde{b}^R$ is a first order best response for all $p \geq b^R(q_r)$ because the definition of first order optimality is point-wise.

We now show that $\tilde{b}^R$ can be specified on $(q_\ell, q_r)$ so that (i) the probability that $q \in (q_\ell, q_r]$ is lower under $\tilde{b}^R$ than under $b^R$ (thus the probability that $q > q_r$ is higher under $\tilde{b}^R$ than under $b^R$), (ii) $\tilde{b}^R$ is relatively close to $b^R$, and (iii) the conditional revenue under $\tilde{b}^R$, given $q \in (q_\ell, q_r]$, is not significantly below the conditional revenue under $b^R$. First, for a distribution $F$ let $\Delta F \equiv F(\tilde{b}^R(q_r)) - F(b^R(q_r))$; since $\tilde{b}^R$ is first-order optimal and is strictly decreasing on $[q_\ell, q_r]$,

$$
\Delta \tilde{F}^R = \left[ \exp \left( \int_{\tilde{b}^R(q_r)}^{b^R(q_r)} \frac{1}{v(\tilde{\varphi}^R(y)) - y} dy \right) - 1 \right] \tilde{F}^R \left( \tilde{b}^R(q_r) \right)
< \left[ \exp \left( \ln \left[ v(q_r) - \tilde{\varphi}^R(q_r) \right] - \ln \left[ v(q_r) - b^R(q_r) \right] \right) - 1 \right] \tilde{F}^R \left( \tilde{b}^R(q_r) \right)
= \left( \frac{\tilde{b}^R(q_r) - b^R(q_r)}{v(q_r) - b^R(q_r)} \right) \tilde{F}^R \left( \tilde{b}^R(q_r) \right) = \left( \frac{\tilde{F}^R \left( \tilde{b}^R(q_r) \right)}{\tilde{F}^R \left( b^R(q_r) \right)} \right) \Delta F^R. \tag{22}
$$

The first equality follows from the fact that $v$ and $\tilde{\varphi}^R$ are strictly decreasing, and the final equality follows from the fact that $b^R$ is first-order optimal with respect to $F^R$ and is flat on $[q_\ell, q_r]$. Now suppose that $\tilde{F}^R(\tilde{b}^R(q_r)) < F^R(\tilde{b}^R(q_r))$; by inequality (22) it must be that $\Delta \tilde{F}^R < \Delta F^R$, and since $\tilde{F}^R(\tilde{b}^R(q_r)) = F^R(\tilde{b}^R(q_r))$ it follows that $\tilde{F}^R(\tilde{b}^R(q_r)) > F^R(\tilde{b}^R(q_r))$, a contradiction. Then $\tilde{F}^R(\tilde{b}^R(q_r)) \geq F^R(\tilde{b}^R(q_r))$, implying directly that $\Delta \tilde{F}^R \leq \Delta F^R$. Thus point (i) holds for any $\tilde{b}^R$.

Points (ii) and (iii) are shown by construction. For $\delta > 0$ sufficiently small, let $\tilde{b}^R(q_r - \delta) > \tilde{b}^R(q_r) - \delta$. Since $\tilde{F}^R|_{p > \tilde{b}^R(q_r)} = F^R|_{p > \tilde{b}^R(q_r)}$, the expected revenue generated by bid $\tilde{b}^R$ under distribution $\tilde{F}^R$, conditional on $p > \tilde{b}^R(q_r)$, is identical to the expected revenue generated by bid $b^R$ under distribution $F^R$, conditional on $p > \tilde{b}^R(q_r)$. Letting $\tilde{b}^R|_{p < \tilde{b}^R(q_r)} = b^R|_{p < \tilde{b}^R(q_r)}$, we have $||\tilde{b}^R - b^R|| < (q_r - q_\ell)\delta + (\tilde{b}^R(q_r) - b^R(q_r))\delta$ by construction. By point (i) and the analysis in the proof of Lemma 15, $\tilde{F}^R|_{p < \tilde{b}^R(q_r)} \leq_{\text{FOSD}} F^R|_{p < \tilde{b}^R(q_r)}$, and so the expected revenue generated by bid $\tilde{b}^R$ under distribution $\tilde{F}^R$, conditional on $p < \tilde{b}^R(q_r)$, is $O(\delta)$ lower than the expected revenue generated by bid $b^R$ under distribution $F^R$, conditional on $p < \tilde{b}^R(q_r)$. Finally, the utility lost when $p \in [\tilde{b}^R(q_r), \tilde{b}^R(q_r)]$ may be bounded in the following
way. When \( p \in [\tilde{b}^R(q_r), \tilde{b}^R(q_r) - \delta] \) at most quantity \( \delta \) is lost (versus bid \( b^R \)), with marginal utility at most \( \overline{v} \); this loss is incurred with at most probability 1, so this loss is bounded above by \( \overline{v} \delta \). When \( p \in [\tilde{b}^R(q_\ell) - \delta, \tilde{b}^R(q_\ell)] \), the quantity lost (versus bid \( b^R \)) is at most \( (q_r - q_\ell) < Q \), with marginal utility at most \( \overline{v} \). However, the probability that this quantity is lost is bounded by

\[
\tilde{F}^R(\tilde{b}^R(q_\ell)) - \tilde{F}^R(\tilde{b}^R(q_\ell) - \delta)
\leq \left[ \exp \left( \int_{\tilde{b}^R(q_\ell) - \delta}^{\tilde{b}^R(q_\ell)} \frac{1}{v(\tilde{\phi}^R(y)) - y} \, dy \right) - 1 \right] \tilde{F}^R(\tilde{b}^R(q_\ell) - \delta)
\leq \left[ \exp \left( \int_{\tilde{b}^R(q_\ell) - \delta}^{\tilde{b}^R(q_\ell)} \frac{1}{v(q_\ell) - y} \, dy \right) - 1 \right] \tilde{F}^R(\tilde{b}^R(q_\ell))
\leq \left[ \exp \left( \ln \left[ v(q_\ell) - \tilde{b}^R(q_\ell) - \delta \right] - \ln \left[ v(q_\ell) - \tilde{b}^R(q_\ell) \right] \right) - 1 \right] \tilde{F}^R(\tilde{b}^R(q_\ell))
= \left( \frac{\delta}{v(q_\ell) - \tilde{b}^R(q_\ell)} \right) \tilde{F}^R(\tilde{b}^R(q_\ell)).
\]

Then this probability is also bounded above by a term linear in \( \delta \). Then for any \( \lambda > 0 \) there is \( \delta > 0 \) such that the revenue generated by the first-order optimal bid function \( \tilde{b}^R \) under reserve distribution \( \tilde{F}^R \) is no more than \( \lambda \) below the revenue generated by the first-order optimal bid function \( b^R \) under reserve distribution \( F^R \).

The above two lemmas imply the following approximation result:

**Lemma 17.** Given any best response bid curve \( b(\cdot) \) and any \( \lambda > 0 \), there is a massless reserve distribution \( \tilde{F}^R \) with strictly decreasing first-order best response \( \tilde{b}^R \) such that such that the first order best response to \( F^R \) generates no more than \( \lambda \) less revenue than \( b(\cdot) \).

### G.2.4 An Auxiliary Uniform-Price Auction with Known Values

We maintain the auxiliary assumption that the bidder whose response we analyze has no private information. Having shown that we can restrict attention to random reserve, we continue the analysis by showing that any strictly decreasing first-order optimal bid \( \tilde{b}^R \) generates strictly less revenue than some uniform-price auction (Theorem 18), which we then bound by pay-as-bid revenue in the next and final subsection, where we also drop the no-private-information assumption.

**Lemma 18.** [Uniform-Price Revenue Implementation] Given a massless distribution of reserve prices \( F^R \) and a strictly decreasing first-order optimal bid \( b^R \), there is a distribu-

\[97\] Since \( b(\cdot) < v(\cdot) \) for all units which are received with strictly positive probability (Lemma 6), \( v(q_r) - b^R(q_r) = v(q_r) - \tilde{b}^R(q_\ell) > 0 \).
tion of reserve prices $\hat{F}^R$ such that the uniform-price auction under reserve distribution $\hat{F}^R$ generates the same expected revenue as the pay-as-bid auction with first-order optimal bid $b^R$ and reserve distribution $F^R$.

While the above lemma shows that uniform-price can match the revenue of pay-as-bid, we need to bear in mind that it is an auxiliary result in which we assumed that the bidder analyzed has no private information.

**Proof.** We may assume that the support of the distribution $F^R$ is contained in the support of marginal values on units the bidder can win. Indeed, our assumptions on the utility imply that this support is convex and thus reserves outside of support are either above or below it. The mass of reserve prices above the support can be arbitrarily shifted to reserves in the support, increasing expected revenue. The mass of reserves below the support can be shifted to the minimum of the support, again weakly increasing the revenue. The latter operation might create an atom at the bottom of the support, but as we have seen in the proofs for Section 3 (cf. Appendix C.5), this atom does not affect the bidder’s best response behavior. Under these assumptions, truthful reporting, $b \equiv v$, is the essentially unique equilibrium in a uniform-price auction with random reserve drawn from $F^R$. Under a random reserve distribution, each bidder’s problem is a single-person decision problem. Because demand at a particular price does not affect outcomes at other prices, at each price bidders should demand a utility-maximizing quantity. Thus at each $p$, $v(\varphi^R(p)) = p$.\(^{98}\)

Revenue in the pay-as-bid auction under reserve distribution $F^R$ is

$$E[\pi] = \int_b^v \left( p \varphi^R(p) + \int_p^v \varphi^R(x) \, dx \right) f^R(p) \, dp.$$ 

Define $\hat{F}^R$ so that

$$\hat{F}^R \left( v \left( \varphi^R(p) \right) \right) = F^R \left( p, s \right).$$

By construction, $\hat{F}_p^R(v(\varphi^R(p)))v_q(\varphi^R(p))\varphi_p^R(p) = F_p^R(p)$. Additionally, Supp $\hat{F}^R = [\underline{p}, \overline{v}]$, and in a uniform-price auction with reserve distribution $\hat{F}^R$, it is weakly optimal for the bidder to submit truthful bids for all quantities $q$ such that $v(q) \in [b, v]$. The revenue in this auction is

$$E[\hat{\pi}] = \int_b^v pv^{-1}(p) \hat{F}_p^R(p) \, dp.$$ 

Apply a change of variables, so that $p = \hat{v}(\varphi^R(p'))$. Then $dp = v_q(\varphi^R(p'))\varphi_p^R(p') \, dp'$. Since $b$ is strictly decreasing and first-order optimal, $\varphi$ and $\varphi_p$ are well-defined for all feasible prices $p$.\(^{98}\)
\[ \varphi^R(p) = 0, \] this gives
\[
\mathbb{E}[\hat{\pi}] = \int_b^\bar{b} v(\varphi^R(p')) v^{-1}\left( v\left( \varphi^R(p') \right) \right) \hat{F}^R_p\left( v\left( \varphi^R(p') \right) \right) v_q\left( \varphi^R(p') \right) \varphi^R(p') dp'.
\]

Then compare,
\[
\mathbb{E}[\pi] - \mathbb{E}[\hat{\pi}] = \int_b^\bar{b} \left( p\varphi^R(p) + \int_p^\bar{b} \varphi^R(x) dx \right) F^R_p(p) - v\left( \varphi^R(p) \right) \varphi^R(p) F^R(p) dp
= \int_b^\bar{b} \left( - v\left( \varphi^R(p) \right) - p \right) \varphi^R(p) + \int_p^\bar{b} \varphi^R(x) dx \right) F^R_p(p) dp
= \int_b^\bar{b} \left( - \left[ \frac{F^R(p)}{F^R_p(p)} \right] \varphi^R(p) + \int_p^\bar{b} \varphi^R(x) dx \right) F^R_p(p) dp
= - \int_b^\bar{b} \varphi^R(p) F^R(p) dp + \int_b^\bar{b} \int_p^\bar{b} \varphi^R(x) dx F^R_p(p) dp
= - \int_b^\bar{b} \varphi^R(p) F^R(p) dp + \left[ \int_p^\bar{b} \varphi^R(x) dx F^R(p) \right]_{p=b}^{\bar{b}} + \int_b^\bar{b} q^R(p) F^R(p) dp = 0.
\]

The transition from the second line to the third comes from the bidder’s first-order condition under random reserve. Then the uniform-price auction with reserve distribution \( \hat{F}^R \) generates the same revenue as the pay-as-bid auction with reserve distribution \( F^R \) and first-order optimal bid \( b^R \).

\[ \square \]

### G.2.5 Revenue Dominance of Deterministic Mechanisms with Known Values

Our previous lemmas imply that, when a bidder has no private information, the seller can weakly improve the revenue obtained from this bidder by implementing a uniform price auction with a random reserve price. These results are independent of opponent strategies in the pay-as-bid auction. Furthermore, we argued above that when the bidder participates in an auction with a random reserve price (and sufficiently large fixed supply) her best response is independent of her opponents’ strategies. Thus, if the seller knew each bidder’s private information, he could improve his revenue by implementing a bidder-specific uniform price auction with a random reserve price.

We are now ready to conclude the proof of Lemma 2 by showing that the above uniform price auction generates less revenue than a deterministic pay-as-bid auction, still in the auxiliary environment in which bidders have no private information, or as we may also express it, when their information is known to the seller.
Proof. Focusing on one bidder and putting together Lemmas 15, 16, and 18 we can conclude that for any $\lambda > 0$ and any random elastic supply in a pay-as-bid auction, there is a uniform-price auction with random reserve that raises from the bidder we focus on at least the pay-as-bid auction revenue minus $\lambda$. As we have seen in the first paragraph of the proof of Lemma 18, in this uniform-price auction we may assume that the bidder bids his or her marginal values (at all prices in the support of the random reserve distribution), and ex post revenue is always weakly below monopoly revenue. It follows that the uniform-price auction’s revenue is maximized by selling the deterministic monopoly quantity with an appropriate reserve price. By Theorem 5, this revenue is equivalent to what the seller would obtain by implementing a pay-as-bid auction for the (deterministic) monopoly quantity, with or without a reserve price. Thus, to maximize the revenue obtained from a single bidder whose information is known to the seller, it is optimal to deterministically sell the bidder the monopoly quantity.

Because bidders are symmetric, it follows that it is optimal to deterministically sell them the aggregate monopoly quantity (note that the equilibrium price will be the monopoly price as long as the seller sets the reserves weakly below it).

G.3 Proof of Theorem 10 (Optimality of Deterministic Mechanisms)

Proof. If the seller knows the bidders’ common signal $s$, the optimal quantity in a pay-as-bid auction is $Q^*(s) \in \arg \max_{Q \leq Q^{\max}} Q \hat{v}(Q; s)$, and in the unique equilibrium of this pay-as-bid auction, $p^*(Q^*(s); s) = \hat{v}(Q^*(s); s)$. Let $Q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a supply curve, where $Q(p) = \inf \{Q^*(s) : p^*(s) > p\}$. Bidder values are regular, so $Q$ is increasing. Then equilibrium in the pay-as-bid auction with supply curve $Q$ is such that for any bidder signal $s$, $p(Q^*(s); s) = \hat{v}(Q^*(s); s)$, and revenue is maximized for each type independently.

G.4 Proof of Theorem 11 (Revenue Dominance of Pay-as-Bid)

Proof. Consider the (deterministic) optimal supply curve derived in Theorem 10. Given this supply curve, there is an equilibrium of the uniform-price auction in which bidders submit truthful bids. As in the pay-as-bid auction, for any realization of the bidder-common signal $s$ the market clearing price and quantity corresponds to the monopoly solution, and revenue in this equilibrium of the uniform-price auction is equivalent to revenue in the unique equilibrium of the optimal pay-as-bid auction. No higher revenue is feasible in the uniform-price auction—even with different distribution over supply-reserve—because for known $s$ the

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Footnote: Equilibrium uniqueness is established in Theorem 9.
revenue is bounded above by monopoly revenue.