A Case for Pay-as-Bid Auctions

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August 2023

Abstract

Pay-as-bid (or discriminatory) auctions are frequently used to sell homogenous goods such as treasury securities and commodities. We prove the uniqueness of their pure-strategy Bayesian Nash equilibrium and establish a tractable representation of equilibrium bids. Building on these results we analyze the optimal design of pay-as-bid auctions, as well as uniform-price auctions (the main alternative auction format). We show that supply transparency and full disclosure are optimal in pay-as-bid, though not necessarily in uniform-price; pay-as-bid is revenue dominant and might be welfare dominant; and we provide an explanation for the revenue equivalence observed in empirical work.

First draft: 2018. Section 4 subsumes Pycia and Woodward "Pay-As-Bid: Selling Divisible Goods" (2014). We would like to thank Pär Holmberg as well as John Asker, Susan Athey, Lawrence Ausubel, Gary Biglaiser, Sushil Bikhchandani, Simon Board, Ben Brooks, Eddie Dekel, Jeffrey Ely, Christian Ewerhart, Natalia Fabra, Amanda Friedenberg, Rod Garratt, Hugo Hopenhayn, Ali Hortaçsu, Emir Kamenica, Jakub Kastl, Paul Klemperer, Shengwu Li, Albert Ma, Daniel Marszalec, David McAdams, Debasis Mishra, John Moore, Piotr Mucha, Ichiro Obara, Krzysztof Oleszkiewicz, Marco Pagnozzi, Andrea Pratt, Phil Reny, Jozsef Sakovics, Ali Shourideh, Guofu Tan, Ina Taneva, Alex Teytelboym, Xavier Vives, Pierre-Olivier Weill, Simon Wilkie, Milena Wittwer, Andrew Yates, Hisayuki Yoshimoto, and audiences at UCLA, USC, SWET, ISMP, Econometric Society North American Summer Meetings, EC, the Lund Market Design Conference, U Edinburgh, U Zurich, Caltech, Carnegie Mellon/U Pittsburgh, SET Seminars in Economic Theory, Econometric Society European Winter Meetings, Stanford Market Design, Swiss Theory Day, EC'21, Michigan State, Oxford, Stanford, ESSET'22, and U Chicago for their comments. Pycia gratefully acknowledges the support of the UCLA Department of Economics.

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1 Introduction

Each year, securities and commodities worth trillions of dollars are allocated through multi-unit auctions. Pay-as-bid is one of two main auction formats for these sales, the other format being uniform-price. Pay-as-bid is often used to sell treasury securities and to distribute electricity generation. It is also used in government operations such as large-scale asset purchases in the U.S. (quantitative easing), and it is implicitly run in financial markets when limit orders are followed by a market order.[1]

Despite their economic importance, relatively little is known about equilibrium behavior in pay-as-bid auctions. Accordingly, little is known about the design problem faced by the pay-as-bid auctioneer: for instance, what is the optimal reserve price, and how does transparency about supply affect the seller's revenue? Furthermore, what explains the rough revenue equivalence of pay-as-bid and uniform-price auctions found in empirical work?[2]

This paper addresses these open questions in environments in which the bidders are symmetrically informed, an assumption that is approximately satisfied in many multi-unit auction environments.[3] For example, the value of a treasury security can be inferred from the prices of its close substitutes and from the forward contracts on the current issue traded ahead of the auction. The U.K. Debt Management Office highlights this feature of the informational environment in which it sells British gilt-edged securities, noting in its guide that:

> "There are often similar gilts already in the market to allow ease of pricing [...] This suggests that bidders are not significantly deterred from participation by not knowing what the rest of the market's valuation of the gilts on offer is" [UK DMO, 2012].

In empirical analyses, Hortaçsu et al. [2018] argue that bidders in U.S. Treasury auctions of short-term securities are nearly symmetrically informed, Armantier and Lafhel [2009] argue that bidders in Bank of Canada auctions are essentially symmetric, and Hattori and

[1]Pay-as-bid auctions are also referred to as discriminatory, or multiple-price auctions. OECD [2021] finds that 25 of 37 countries surveyed allocate securities via pay-as-bid auctions; Brenner et al. [2009] find that 33 of 48 countries surveyed use pay-as-bid. Del Río [2017] finds that 27 of 31 markets surveyed distribute electricity generation via pay-as-bid auction (see also Maurer and Barroso [2011]). In all these studies most of the remaining markets are cleared by uniform-price auction and in some markets both formats are used. For financial markets, see, e.g., Glosten [1994].

[2]Pay-as-bid auction equilibria have been constructed in parameterized environments; see our discussion below. The empirical literature on multi-unit auctions provides no definitive result on which auction format raises more revenue; Hortaçsu et al. [2018] posit that this is potentially because bidders retain little surplus.

[3]Our main results are robust to the presence of small informational asymmetries, see our Conclusion for a discussion.

Takahashi [2022] argue the same for bidders for Japanese government bonds.[4] While our assumptions are borne out in some important multi-unit auctions, they are not satisfied in others: for example, Armantier and Sbaï [2009] argue that bidders in French debt auctions are asymmetrically informed, and Cole et al. [2022] argue that in Mexican treasury auctions some bidders are informed (and have virtually identical information) while other bidders are uninformed.

Although we assume bidders have symmetric information, our results allow any informational asymmetry between the seller and the bidders. The difference between the seller's and the bidders' information is typical of the problem we study because the seller designs the auction before—usually substantially before—the bidders submit their bids; the seller may also want to set a single design for multiple auctions. We allow for uncertainty of the total supply available for auction as exogenous supply uncertainty is a feature of some securities auctions, e.g. in the United States [TreasuryDirect, 2022] and Japan [Hattori and Takahashi, 2022].[5] We allow an arbitrary number of bidders and general demands, and thus provide a substantively more general treatment than previous analyses, which relied on either large markets or strong parametric assumptions (cf. Swinkels [2001], Ausubel et al. [2014], and the discussion below).

A starting point for our analysis of equilibria is Theorem 1, which determines the lowest equilibrium market-clearing price. This price bound and our subsequent design insights are valid whether or not we allow mixed strategy-equilibria (cf. Appendix A), but our theory of equilibrium bidding in pay-as-bid auctions focuses on pure-strategy equilibria. We prove that pure-strategy equilibrium is unique (Theorem 2), in contrast with the substantial equilibrium multiplicity present in uniform-price auctions [Wilson, 1979, Klemperer and Meyer, 1989, Wang and Zender, 2002].[6] In this unique equilibrium, each bidder responds to stochastic

[4] In addition, our result that in absence of substantive uncertainty bids in the pay-as-bid auction are approximately flat, provides a test of the symmetric information assumption. Another natural test of the symmetry assumption is the difference between auction price and the subsequent secondary market price. Bid flatness and small primary- and secondary-market price differences has been observed in treasury auctions in several countries, see Section 7 for a discussion.

[5] We discuss the exogenous randomness in more detail in Section 5. In the context of securities, U.S. Treasury auction regulations do not provide for the announcement of noncompetitive demand prior to the submission of competitive bids [Garrison et al.], and in Swiss Treasury auctions supply is not announced [Ranaldo and Rossi, 2016]. An analogous uncertainty over auctioned demand is a feature of many spot electricity auctions, where demand is determined by the current state of electricity usage; cf. Federico and Rahman [2003], Hortaçsu and Puller [2008], and U.S. Federal Energy Regulatory Commission [2020], among others.

[6] Uniqueness plays a major role in empirical studies of pay-as-bid auctions. Estimation strategies based on the first-order conditions, or the Euler equation, rely on agents playing comparable equilibria across auctions in the data (Février et al. [2002], Hortaçsu and McAdams [2010], Hortaçsu and Kastl [2012], and Cassola et al. [2013]). Equilibrium uniqueness plays an even larger role in the study of counterfactuals (see, e.g., Armantier and Sbaï [2006] and Armantier and Sbaï [2009]).

residual supply (that is, the supply given the bids of the remaining bidders), and a best response picks points on the realizations of residual supply. In determining a best response, the bidder needs to keep in mind that, in a pay-as-bid auction, a bid is paid not only when it is marginal (at the clearing price) but also whenever it is strictly above the clearing price. We show that despite these subtleties the equilibrium bids have an unexpectedly tractable closed-form representation: the bid for a unit is a weighted average of marginal values on this and higher units (Theorem 3). We also establish a sufficient condition for the existence of equilibrium (Theorem 4); our condition is satisfied when, e.g., there are sufficiently many bidders and their marginal values are smooth.

Turning to design questions, we establish the seller maximizes revenue by transparently setting the auction's aggregate supply.[7] Specifically, revenue in the unique pure-strategy equilibrium is maximized when supply is deterministic (Theorem 5). Thus determining the optimal supply distribution is equivalent to the simpler problem of a monopolist who sets a price and a quantity cap.[8] In some treasury auctions—e.g. in U.S. uniform-price auctions and Japanese pay-as-bid auctions (cf. Section 5.2)—the distribution of supply is partially determined by the demand from non-competitive bidders, and treasuries and central banks retain only partial ability to influence supply distributions but may have pertinent supply information prior to the auction. We therefore also address the question of how much data on non-competitive bids a revenue-maximizing seller should release, and show that the seller wants to commit to fully reveal the realization of supply prior to soliciting bids (Theorem 6).[9] These principles of transparent design simplify the design of pay-as-bid auctions in a way that does not carry over to the design of uniform-price auctions; for the latter we show that neither deterministic supply nor information release are necessarily revenue-optimal (Lemma 1).[10]

[7] We focus on sellers whose objective is revenue maximization. For example, the U.K. Debt Management Office's primary objective in security auctions is, "to minimise over the long term, the costs of meeting the Government's financing needs," and the U.S. Treasury's primary objective in security auctions is, "to finance the government at the lowest cost over time." [United Kingdom Debt Management Office, 2012, U.S. Department of the Treasury, 2019].

[8] In the main text we focus on the seller setting reserve price and distribution of supply in pay as bid; in Appendix A we show that our insights are valid for sellers setting a distribution over elastic supply curves provided bidders' values satisfy a Myerson-like regularity assumption.

[9] For the optimality of revealing other relevant information, cf. our supplementary note, Pycia and Woodward [2023a].

[10] The reason is the multiplicity of equilibria in uniform-price auctions. Specifically, these auctions admit equilibria with a wide range of revenues; see, e.g., Kremer and Nyborg [2004], LiCalzi and Pavan [2005], McAdams [2007], Burkett and Woodward [2020b], and Marszalec et al. [2020]. Depending on the auctioneer's concern about equilibrium selection, anticipated revenue may improve with some randomization, see our Lemma 1. Equilibrium in the optimally designed uniform-price auction becomes unique (and revenue-equivalent to pay-as-bid) if the seller knows the bidders' information; cf. Corollary 6. The empirical impact of transparency has been extensively studied in the context of over-the-counter markets; for a recent review

Leveraging our design results, we are able to compare revenues in optimally-designed pay-as-bid and uniform-price auctions. We prove that the pay-as-bid format always raises weakly higher revenue than the uniform-price format (Theorem 7).[11] In effect, a revenue-maximizing seller would run a uniform-price auction only if its revenue equaled that of pay as bid; we may thus expect counterfactual analysis from uniform-price auctions chosen by revenue-maximizing sellers to find approximate revenue equivalence. Another reason for the revenue equivalence to obtain is if bidders in uniform-price bid truthfully for the marginal unit, a semi-truthful strategy assumed by some major empirical studies comparing revenues between pay-as-bid and uniform-price auctions.[12] In this way, our results provide a theoretical explanation for the approximate revenue equivalence found in the empirical literature we discuss in Section 7.

Our design analysis is focused on pay-as-bid and uniform-price auctions as these are the two formats treasuries typically choose between. In principle, other mechanisms are possible. For example, the correlation present in environments we study enables surplus extraction mechanisms proposed by Myerson [1981] and Crémer and McLean [1988] in which bidders are induced to reveal their valuations by being charged for differences in their reports, thus allowing the auctioneer to charge prices extracting nearly all surplus; such mechanisms are sensitive to collusion and not observed in practice. Or, the government, which has access to similar macroeconomic data as the bidders, might estimate and post optimal prices. Prior to the Great Depression, fixed-price mechanisms were employed by, for instance, the U.S. Treasury, and led to problems such as regular over-subscription, indicating that prices were set too low.[13] A common economic explanation of such government underpricing problems is the capture of policy-makers by bank lobbies, cf. Buchanan et al. [1980], Laffont and Tirole [1993] and Dal Bó [2006]. Competitive auctions help the auctioneer avoid such underpricing problems.[14]

of this literature see e.g. Garratt et al. [2019]. The impact of transparency in uniform-price auctions has been experimentally studied by Hefti et al. [2019].

[11] This revenue comparison extends to any deterministic distribution of supply, with the same proof, provided supply is identical in both auctions. The welfare comparison depends on the environment and equilibrium selection in uniform price.

[12] See e.g. Hortaçsu and McAdams [2010] and Marszalec [2017], and our discussion below. Bidding truthfully for the marginal unit can be—but does not need to be—supported in an equilibrium of an optimally-designed uniform-price auction. Bids that are robust to informational uncertainty, an equilibrium selection inspired by Klemperer and Meyer [1989], are not semi-truthful in this sense, cf. Appendix G.1.

[13] Garbade [2008] provides an overview, but stops short of explaining the reasons for the low prices.

[14] In a symmetric-information environment, the auctioneer could also try to extract all bidder surplus by (for example) holding a first-price auction for the entire aggregate quantity and then allowing the winner to subdivide and resell the awarded allocation. However, market cornering has proved problematic in treasury auctions [Jegadeesh, 1993], and such “all-or-nothing” mechanisms are therefore politically infeasible. Similar arguments may be posed against many other exotic and nonstandard allocation mechanisms. The general divisible-good revenue maximization question was addressed by Maskin and Riley [1989], whose optimal

In total, our results make a case in favor of implementing pay-as-bid over uniform-price. Our model is stylized, and there are many aspects of real-world auctions it fails to capture, e.g., term structure [Klemperer, 2010], bidder asymmetry [Armantier and Sbaï, 2009, Cole et al., 2022, Pycia and Woodward, 2023b], restrictions on permissible bids [Kastl, 2012], pre-auction investments (including information acquisition) [Bergemann and Välimäki, 2002, Arozamena and Cantillon, 2004, Gershkov et al., 2021], entry [Bulow and Klemperer, 1996, Allen et al., 2022], reputational incentives [Marszalec et al., 2020], and distribution of rents [Pycia and Woodward, 2023b]. Nonetheless, we show that pay-as-bid has substantive advantages over uniform-price that have not been previously recognized.

1.1 Literature

Our bound on equilibrium prices is the first such bound that applies to all pure-strategy equilibria, as well as the first such bound that allows for mixed-strategy equilibria. The special cases of our bound are implicit in the equilibrium constructions in the parametric examples of pay-as-bid we discuss below.[15]

There is a large literature on equilibrium existence in pay-as-bid auctions. Linear equilibria have been constructed in the linear-Pareto examples we discuss below. In our more general symmetric-information environment, Holmberg [2009] proves the existence of equilibrium when the distribution of supply has a decreasing hazard rate, and recognizes the possibility that pure-strategy equilibrium may not exist.[16] Our sufficient condition for existence encompasses Holmberg's prior conditions and is substantially milder.[17]

Uniqueness was studied by Wang and Zender [2002], who prove the uniqueness of symmetric equilibria in which bids are piecewise continuously-differentiable functions of quantities and supply is invertible from equilibrium prices under strong parametric assumptions of linear utilities and unbounded Pareto distributions. In a linear-Pareto environment in which

mechanism is quite complex.

[15] A different bound, in terms of competitive markets, was obtained by Swinkels [1999] for large economies. Our bound is valid in all finite markets.

[16] See Genc [2009] and Anderson et al. [2013] for discussions of potential problems with equilibrium existence.

[17] In asymmetric information settings, Athey [2001], McAdams [2003], and Reny [2011] have shown that equilibrium exists in multi-unit (discrete) pay-as-bid auctions, and Woodward [2019] established existence in the asymmetric-information analogue of the divisible-good model we study. A key difference between these papers and ours is that the presence of private information allows the purification of mixed-strategy equilibria; such purification is not possible in the symmetric-information setting. For equilibrium existence in multi-unit auctions, see also Břeský [1999], Jackson et al. [2002], Reny and Zamir [2004], Jackson and Swinkels [2005], McAdams [2006], Armantier et al. [2008], Břeský [2008], and Kastl [2012]. Milgrom and Weber [1985] show existence of mixed-strategy equilibria.

the maximum supply strictly exceeds the maximum total quantity the bidders are willing to buy, Holmberg [2009] proved the uniqueness of symmetric equilibria in which bid functions are twice differentiable.[18] Ewerhart et al. [2010] and Ausubel et al. [2014] independently expand these analyses to Pareto supply with bounded support and show the uniqueness of equilibria in which bids are linear functions of quantities. In contrast, we look at all Bayesian Nash equilibria of our model, we impose no parametric assumptions, and we do not require that some part of the supply is not wanted by any bidder.[19]

Our uniqueness result is also related to Klemperer and Meyer [1989] who established uniqueness in a duopoly model closely related to uniform-price auctions: when two symmetric and uninformed firms face random demand with unbounded support, then there is a unique equilibrium in their model.[20] The main difference between the two papers is, of course, that Klemperer and Meyer analyze the uniform-price format, while we look at pay-as-bid.[21]

Our bid representation theorem may be seen as a finite-market counterpart of Swinkels [2001], who studies pay-as-bid and uniform-price auctions in large markets; in the limit, as the number of bidders goes to infinity, our representations are equivalent. He restricts attention to equilibria that are asymptotically environmentally similar, an assumption we do not impose. Our contribution also lies in establishing the representation of bids as averages of marginal values in all finite markets and not only in the limit. Holmberg [2009] derives a closed-form representation for symmetric and smooth equilibria subject to constraints on supply. We make no such assumptions, and instead prove that equilibria are symmetric and smooth; our results therefore provide support for his analysis and our finite-market representation of bids as weighted averages of marginal values is new.

[18]Holmberg's assumption that bidders do not want to buy part of the supply represents a physical constraint in the reverse pay-as-bid electricity auction he studies: in his paper bidders supply electricity and face capacity constraints, and beyond a certain level they cannot produce more. This low-capacity assumption drives the analysis and it precludes directly applying the same model in the context of securities auctions in which bidders are always willing to buy more (provided the price is sufficiently low).

[19]As a consequence of this generality, we need to develop a methodological approach which differs from that of the prior literature. McAdams [2002] and Ausubel et al. [2014] have also established the uniqueness of equilibrium in their respective parametric examples with two bidders and two goods.

[20]The analogue of their unbounded support assumption is our assumption that the support of supply extends all the way to no supply. While the two assumptions look analogous they have very different practical implications. In a treasury auction, for example, a seller can guarantee that with some small probability the supply will be lower than the target; in fact, in practice the supply is often random and our support assumption is satisfied. On the other hand, it is substantially more difficult, and practically impossible, for the seller to guarantee the risk of arbitrarily-large supplies.

[21]The literature has known since Wilson [1979] that the uniform-price auction may admit multiple equilibria. No similar multiplicity constructions exist for pay-as-bid auctions (recently Cole et al. [2017] have shown that pay-as-bid may induce multiplicity of equilibrium levels of information acquisition prior to the auction, though not in the auction itself). The proof of our uniqueness result follows a differential analysis familiar from uniqueness results for first-price auctions (see, e.g., Lizzeri and Persico [2000], Maskin and Riley [2003], and Lebrun [2006]), but our analysis establishing the initial condition for the differential analysis is distinct.

Our bid representation is surprising in the context of prior finite-market literature, which can be naturally read as suggesting that pay-as-bid equilibria are complex. Prior constructions of finite-market equilibria focused on the setting in which bidders' marginal values are linear in quantity and the distribution of supply is some instance of the generalized Pareto distribution; see Wang and Zender [2002], Federico and Rahman [2003], Hästö and Holmberg [2006], Holmberg [2009], Ewerhart et al. [2010], and Ausubel et al. [2014].[22] This literature expressed equilibrium bids in terms of the intercept and slope of the linear demand and the parameters of the generalized Pareto distribution. Our general treatment avoids the complexity inherent in expressing bids in terms of parameters of the functional forms studied in the earlier literature.[23]

Our transparency result—that deterministic selling strategies are optimal—may appear familiar from the no-haggling theorem of Riley and Zeckhauser [1983]. However, in multi-object settings the reverse has been shown by Pycia [2006] and Manelli and Vincent [2006]; and, as we show in our Lemma 1, nondeterministic supply may have a role in uniform-price auctions. Furthermore, there is a subtlety specific to pay-as-bid that might suggest a role for randomization: by randomizing supply below the monopoly quantity, the seller forces bidders to compete and bid more for these quantities, and in pay-as-bid the seller collects the raised bids even when the realized supply is near the monopoly quantity. We show that, despite these considerations, committing to deterministic supply is indeed optimal.[24]

Our full disclosure result may at first glance appear to be a consequence of Milgrom and Weber's [1982] celebrated linkage principle; the linkage principle is however known to fail in the multi-unit auction context (cf. Perry and Reny [1999] and Vives [2010]) and our disclosure result relies instead on our bound on revenues in pay-as-bid auctions with random supply. Furthermore, while our setting is one of Bayesian persuasion and information design, the full disclosure we establish stands in stark contrast to Kamenica and Gentzkow's [2011] paradigmatic insight that in information design and Bayesian persuasion the sender wants to withhold—or obfuscate—information. Related to information design, Bergemann et al.

[22] We focus our discussion on settings with decreasing marginal utilities; for constant marginal utilities see Back and Zender [1993] and Ausubel et al. [2014].

[23] The difficulty in constructing an equilibrium in pay as bid is two-fold. In equilibrium, each bidder responds to the stochastic residual supply (that is, the supply given the bids of the remaining bidders) and in determining her best response, a bidder needs to keep in mind that: (i) A bid that is marginal if a particular residual supply curve is realized is paid not only when it is marginal, but also in any other state of nature that results in a larger allocation, and hence the bidder faces tradeoffs across these different states of nature; and (ii) Bid curves need to be weakly decreasing in quantity, potentially a binding constraint.

[24] Recently Chen et al. [2019] show that individual outcomes of a given random mechanism can be replicated by a deterministic mechanism when there are multiple privately informed participants, while we show that not only can the maximal revenue generated by any random pay-as-bid auction be obtained by some deterministic mechanism, but also that this is possible without fundamentally changing the auction mechanism.

[2017] and Bergemann et al. [2019] also find the optimality of withholding information in single-unit auctions. This shows that our full disclosure result crucially relies on the specifics of the pay-as-bid format and the divisibility of the good sold.[25]

As discussed above the relevant design questions are the choice between the two standard auction mechanisms, pay-as-bid and uniform-price, and the design of their reserve, supply, and information. Design issues have been addressed in the context of uniform-price auction. The design analysis of uniform-price has focused on preventing collusive equilibria: Fabra [2003] and Marszalec et al. [2020] show that collusion is easier in uniform price than in pay as bid; however Klemperer and Meyer [1989] point out that the auctioneer can induce competition in a uniform-price auction by introducing slight randomness in supply, Kremer and Nyborg [2004] look at the role of tie-breaking rules, LiCalzi and Pavan [2005] and Burkett and Woodward [2020b] at elastic supply, McAdams [2007] at commitment, and Burkett and Woodward [2020a] at the role of price selection. By proving equilibrium uniqueness for pay-as-bid we show its resilience to equilibrium collusion, thus providing a pay-as-bid counterpart for this literature.[26] We also contribute to this uniform-price literature directly by showing that not only the seller but also the bidders might be made worse off by the possibility of tacit collusion; the reason is that the seller who expects a collusive equilibrium in uniform-price auction might optimally respond by setting a high reserve price, thus recovering some of the revenue at the cost of bidders' surplus.

Our revenue and welfare comparisons between pay-as-bid and uniform-price auctions contribute to the rich discussion of the pros and cons of these two formats. Swinkels [2001] focused on equilibria satisfying an asymptotic environmental similarity assumption and showed that pay-as-bid and uniform-price are revenue- and welfare-equivalent in large markets; Jackson and Kremer [2006] find revenue- and welfare- equivalence in large market limit under the assumption that the proportion of supply to the number of bidders vanishes to zero; our

[25]In single-unit auctions bidders necessarily have full information regarding the quantity supplied, and the auctioneer's role in information design is inherently limited. Fang and Parreiras [2003] and Board [2009] study the limits of the linkage principle and the resulting benefits of information withdrawal or obfuscation. The optimality of obfuscation generally obtains in setting in which the participation constraints are interim and the seller cannot charge for information (cf. Bergemann and Pesendorfer [2007]). Even if the seller can charge for information, obfuscation is shown to be optimal by Li and Shi [2017] except under orthogonality assumptions of Eső and Szentes [2007]. Obfuscation is also established in other settings in which—like in our auction setting—the sender's interest (more revenue) is fundamentally misaligned with the bidders' interests (reducing payment); in a global games context see, e.g., Li et al. [2023]. For analysis of bidders' investment in information acquisition in auctions see e.g. Persico [2000] who finds that bidders in first-price auctions acquire more value-relevant information than bidders in second-price auctions. Finally, while we study a seller/sender who is able to commit to a disclosure strategy, our disclosure result immediately implies that a sender unable to commit would also fully reveal supply information. For information disclosure under no commitment see e.g. Grossman and Hart [1980] and Milgrom [1981].

[26]Relatedly, the empirical analysis of Häfner [2020] suggests that there is no collusion in the Swiss import permit pay-as-bid auctions.

equivalence result does not rely on the size of the market, nor on an environmental similarity assumption, nor on extreme competition among bidders. Wang and Zender [2002] find pay-as-bid revenue superior in the equilibria of the complete-information linear-Pareto model their consider, and Woodward [2021] extends this dominance to mixed-price combinations of pay-as-bid and uniform-price auctions. Ausubel et al. [2014] show that—with ex-ante asymmetric bidders with flat demands—either format can be revenue superior.[27] In aggregate, prior theoretical work on the pay-as-bid versus uniform-price question has focused on revenue comparisons for fixed supply distributions and has allowed for neither reserve price nor supply optimization; indeed, the previous studies of pay-as-bid auctions with decreasing marginal values employed parametric specifications that did not support the analysis of design questions. Thus the earlier work could not address whether a well-designed pay-as-bid auction is preferable to a well-designed uniform-price auction. We go beyond these earlier papers both by allowing for the seller's optimization and by imposing no assumptions on the seller's information about the bidders.

Our divisible-good optimal revenue equivalence result provides a benchmark for the long-standing empirical debate whether pay-as-bid or uniform-price auctions raise higher expected revenues. This debate has attracted substantial empirical attention, with Hortaçsu and McAdams [2010] and Barbosa et al. [2020] finding no statistically significant differences in revenues, Février et al. [2002], Kang and Puller [2008], Armantier and Lafhel [2009], Marszalec [2017], Mariño and Marszalec [2020], and Hattori and Takahashi [2022] finding slightly higher revenues in pay-as-bid, and Goldreich [2007], Castellanos and Oviedo [2008], Armantier and Sbaï [2006], and Armantier and Sbaï [2009] finding slightly higher revenues in uniform-price. Hortaçsu et al. [2018] argue that the revenues are similar because not much surplus is retained by bidders.

Our results regarding the selection of auction format have other empirical implications. We show in our analysis of the auction design game that the auctioneer either strictly prefers a pay-as-bid auction or is indifferent between the pay-as-bid and uniform-price formats. All else equal, our model suggests that pay-as-bid auctions should be more prevalent than uniform-price auctions. This claim is supported by the multi-country surveys of treasury auctions in Brenner et al. [2009] and OECD [2021], which find that pay-as-bid auctions are implemented by more than twice as many nations as implement uniform-price auctions,

[27]When bidders have symmetric or non-flat demands, pay-as-bid is revenue superior in all examples they consider. The special supply distributions these papers consider are not revenue-maximizing, hence there is no conflict between their strict rankings and our revenue comparisons. See also Jackson and Kremer [2006] and Fabra et al. [2006] who find that—with non-optimized supply—either format can be revenue superior, and Anwar [1999] and Engelbrecht-Wiggans and Kahn [2002] for revenue comparisons with flat demands. Fabra et al. [2011] show that the two formats may lead to the same investments in capacity. Hinz [2004] shows revenue equivalence in multi-unit auctions with single-unit demand.

as well as the analysis of electricity markets in Del Río [2017], which finds that pay-as-bid auctions represent nearly 90% of auctions for allocating the capacity for renewable electricity generation.[28]

2 Example

We preview our results with the following example. Consider n bidders who commonly observe a signal s, drawn uniformly from an interval $[\underline{s}, \overline{s}] \subsetneq (0, +\infty)$; the bidders' marginal values are linear, $v(q; s) = s - \rho q$. Our optimal transparency result (Theorem 5) says that the optimal pay-as-bid auction consists of deterministic supply $Q^\star$ and a reserve price $R^\star$ which solve a classical monopoly problem. In this optimal auction, the equilibrium bids are essentially unique ((2)) and our general bid construction (Theorem 3) takes a particularly simple form: the bids are flat and each bidder receives zero margin on their awarded quantity (cf. also Theorems 1 and (4)). Thus the optimal pay-as-bid auction is determined by

$$Q^{\star\mathrm{PAB}} = \left(\frac{3\overline{s} + \underline{s}}{8\rho}\right) n, \quad R^{\star\mathrm{PAB}} = \frac{\overline{s} + 3\underline{s}}{8}.$$

Determining an optimal uniform-price auction is hampered by equilibrium multiplicity (see our discussion in Section 6), since bidders' choice of equilibrium may depend on the parameterization of the auction. For consistency, we focus on the unique *robust equilibrium* [Pycia and Woodward, 2023a] whose existence is essentially unaffected by perturbations of supply; this approach was pioneered by Klemperer and Meyer [1989] and became the basis for subsequent theoretical literature focusing on environments in which robust equilibria take linear form, cf., e.g., Ausubel et al., 2014. Unlike the pay-as-bid auction, in which optimal supply and reserve operate essentially independently—bidders either receive their demand at the reserve price, or pay their marginal value for the supplied quantity—in a robust equilibrium of the uniform-price auction the reserve price shifts the bids of all bidders, even those who (in equilibrium) pay above the reserve price. We find the optimal uniform-price auction numerically.

As we show in our Theorem 7 and illustrate in Table 1 above, the optimal pay-as-

[28] Among electricity markets surveyed by Maurer and Barroso [2011], about half use pay-as-bid, and most of the remaining markets use uniform price. In their treasury auctions, large countries such as the U.S. often rely on uniform price while smaller countries on pay as bid; this difference is consistent with the example we develop in Section 2, in which the revenue advantage of pay as bid diminishes with the number of bidders. Counterfactual analysis of uniform-price auctions assumes semi-truthful reporting of values to obtain an upper bound on unobserved revenue. Our results suggest caution in interpreting this bound as it is tight only if bidders are playing a seller-optimal equilibrium; otherwise there may be a divergence between observed revenue and counterfactual predictions.

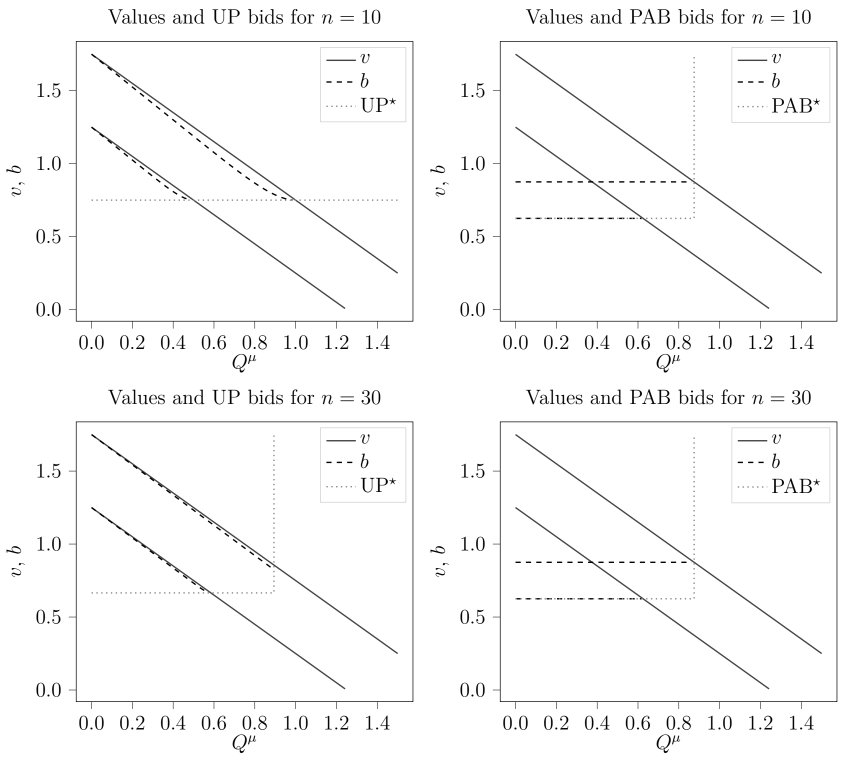

Figure 1: Bids in optimal pay-as-bid and uniform-price auctions, with $n \in \{10, 30\}$ bidders.

n	10	20	30	40
$\mathbb{E}[\pi_{\text{UP}}] / \mathbb{E}[\pi_{\text{PAB}}]$	97.30%	97.30%	97.89%	98.39%

Table 1: Equilibrium expected revenue in the optimal uniform-price auction, as a fraction of expected revenue in the optimal pay-as-bid auction, by number of bidders.

bid auction yields more revenue than the optimal uniform-price auction. This difference goes to zero as the number of bidders goes to infinity (an insight established by Swinkels, 2001), but as the present example shows for realistically-sized markets the difference may be substantial.[29]

3 Model

There are $n \geq 2$ bidders, $i \in \{1, ..., n\}$. Bidder i's marginal valuation for quantity q is denoted $v(q; s)$, where s is a signal known by all bidders but not by the seller. The seller believes that the signal comes from some distribution σ. For any s, we assume that $v(\cdot; s)$ is strictly decreasing where it is strictly positive, and Lipschitz continuous and almost-everywhere differentiable. We impose no assumptions on the space of signals s, except that $v(q; \cdot)$ is integrable for any q. Variability of the common signal s has no strategic importance for bidders participating in an auction, and thus when studying the equilibrium among such bidders in Section 4, we fix s and denote the bidders' marginal valuation by $v(q; s) = v(q)$. Bidders' information plays an important role in the analysis of the seller's problem in Sections 5 and 6. The seller may not know the bidders' information if, for example, the seller needs to commit to the auction mechanism before this information is revealed. Alternatively, the seller may want to fix a single design for multiple auctions.

To simplify the exposition of the design problem, we normalize the seller's cost to 0. Our insights do not hinge on this normalization, and remain valid for any convex increasing cost function.[30] Our design analysis builds on the existence, uniqueness, and bid representation results for equilibria of the pay-as-bid auction. In our equilibrium analysis we assume that aggregate supply Q is drawn from distribution F with support $[0, \overline{Q}]$, and we further assume that F is Lebesgue absolutely continuous on $(0, \overline{Q})$ with continuous density $f > 0$; in all results we also allow F with full mass concentrated at one point. Aggregate supply Q is independent of the bidders' signal s. Otherwise we impose no global assumptions on F.

In our analysis of auction design, the seller is free to choose any distribution F satisfying the above conditions, as long as $\overline{Q} \leq Q^{\max}$, where $Q^{\max}$ is the maximum supply available to the seller.[31] The seller also implements a reserve price $R \geq 0$. While capping aggregate

[29]As Table 2 shows, the countries for which we found this information have between 12 and 35 bidders in their auctions. U.S. Treasury auctions, for example, have roughly 25 bidders; Chinese auctions are an outlier attended on average by 35 bidders. In some contexts there are more bidders, e.g., the largest auctions we found, the 2007 European liquidity auctions, were attended by about 340 bidders.

[30]The reason why more general cost functions do not substantively change the analysis is that it builds on the transparency insight of Theorem 6, and this theorem (and its proof) is valid irrespective of seller's cost function. Our supplementary note Pycia and Woodward [2023a] provides a more detailed discussion.

[31]We could allow for infinite $Q^{\max}$ as long as the optimal monopoly quantity remains finite. This would

supply Q and setting the reserve price R play similar roles in the seller's design problem, our analysis shows that both of these instruments are needed to maximize revenue when the environment is sufficiently rich, e.g., as in the example of Section 2.[32] When the seller employs both of these instruments, the quantity that is allocated is equal to Q if the reserve is not binding, but it may be lower than Q when the reserve price is binding. For any realized quantity $Q \leq \overline{Q}$ and bidders' signal s, denote $Q^R(Q,s) = Q$ for reserve price $R = 0$ and $Q^R(Q,s) = \min\left\{Q, \sum_{i=1,\ldots,n} v^{-1}(R;s)\right\}$ for $R \in (0, v(0;s)]$, where $v^{-1}(\cdot;s)$ is the inverse function of value given bidders' signal s (the inverse is well defined for $R \in (0, v(0;s)]$). Our Theorem 1 below implies that $Q^R(Q,s)$ is the quantity that is actually allocated. In particular, when the reserve is binding, the theorem implies that each bidder receives quantity $Q^R(Q,s)/n = v^{-1}(R;s)$. We use $\overline{Q}^R = Q^R(\overline{Q}, s)$ to denote the effective quantity at the maximum supply $\overline{Q}$.

In the pay-as-bid auction, each bidder submits a weakly decreasing bid function $b^i(q) : [0, \overline{Q}] \to \mathbb{R}_+$. Without loss of generality we may assume that the bid functions are right continuous.[33] The auctioneer then sets the market-clearing price, also known as the stop-out price,

$$p^\star = \max\left\{R, \sup\left\{p' \colon q_1 + ... + q_n \geq Q \text{ for all } q_1, \ldots, q_n \text{ such that } \; b^1(q_1), ..., b^n(q_n) \leq p'\right\}\right\}.$$

If the set over which the supremum is taken is empty, then the stop-out price is set to the reserve price R. Agents are awarded a quantity associated with their demand at the stop-out price,

$$q_i = \max\left\{q' \colon b^i(q') \geq p^\star\right\},$$

as long as there is no need to ration them. When necessary, we ration pro-rata on the margin, the standard tie-breaking rule in divisible-good auctions. The details of the rationing rule have no impact on the analysis of equilibrium bidding.[34] The demand function (the mapping

be so if, e.g., the seller faces increasing and convex marginal costs of supply, or there is no heavy tail of marginal values. In Appendix A, we show—without restrictions to pure strategies, full support, or Lebesgue continuity—that our design insights remain valid if the seller can choose any distribution over elastic supplies.

[32]The relative virtues of regulating prices versus quantities have been studied since Weitzman [1974]. The potential benefit of a hybrid system regulating both prices and quantities was first studied by Roberts and Spence [1976].

[33]This assumption is without loss because we study a perfectly-divisible good and we ration quantities pro-rata on the margin. As the bid function is weakly decreasing, by changing it on measure zero of quantities we can assure the bid function is right continuous. Such a change has no impact on the bidder's profit, or on the profits of any of the other bidders, provided the quantity assigned to each bidder increases when the stop-out price decreases; a monotonicity property satisfied by tie-breaking pro-rata on the margin. In fact, there is no impact on bidders' profits even conditional on any realization of Q.

[34]In equilibrium, supply equals demand at the stop-out price. All we need in our analysis is that rationing rule is monotonic in the sense of footnote 33. The resulting independence of equilibrium of specific tie-

from p to q_i) is denoted by $\varphi^i(\cdot)$. Where $b^i(\cdot)$ is constant, φ^i is not well-defined and we use $\underline{\varphi}^i$ and $\overline{\varphi}^i$ to denote the right- and left-continuous inverses of b, $\underline{\varphi}^i\left(p\right) = \sup\left\{q\colon b^i\left(q\right) > p\right\}$ and $\overline{\varphi}^i\left(p\right) = \sup\left\{q\colon b^i\left(q\right) \geq p\right\}$. Agents pay their bid for each unit received, and utility is quasilinear in monetary transfers; hence,

$$u^i\left(b^i\right) = \int_0^{q^i(p^\star)} v\left(x\right) - b^i\left(x\right) dx.$$

The above formal definition lends itself to the interpretation that bidders submit separate bids for each infinitesimal unit of the good, and the auctioneer first fills the infinitesimal unit with the highest bid, then the infinitesimal unit with the second-highest bid, etc, until the realized supply is allocated or there are no more bids above the reserve price.

Our analysis focuses on pure-strategy Bayesian Nash equilibria, and whenever we write "equilibrium" without any modification we refer to pure-strategy equilibrium. We also include robustness checks for mixed-strategy equilibria, and in all such results we explicitly refer to mixed-strategy equilibria.

4 Pay-as-Bid Equilibrium

We focus on pure strategy equilibria, except as otherwise noted. In the analysis we hold bidders' common signal s fixed and simplify notation by denoting the bidders' marginal valuation $v(q;s)$ by $v\left(q\right)$. We begin the analysis by providing a tight bound on the market price, then we leverage this bound to provide a closed-form expression for the unique equilibrium bid profile.

4.1 Minimum Market Price

Our analysis of optimal bidding relies on the following key theorem in which we allow mixed-strategy equilibria.

Theorem 1. [**Minimum Market Price**] *In any mixed-strategy equilibrium of the pay-as-bid auction, the market clearing price for the effective maximum quantity $\overline{Q}^R$ is, with probability* 1*, given by*

$$p\left(\overline{Q}^R\right) = v\left(\frac{1}{n}\overline{Q}^R\right).$$

As we allow mixed strategies, $p\left(\overline{Q}^R\right)$ is a priori a random variable; part of the theorem's claim is that it is deterministic. Since probability-zero changes to bidding strategies and

breaking rules is in stark contrast to uniform-price auction, where tie-breaking matters; see Kremer and Nyborg [2004].

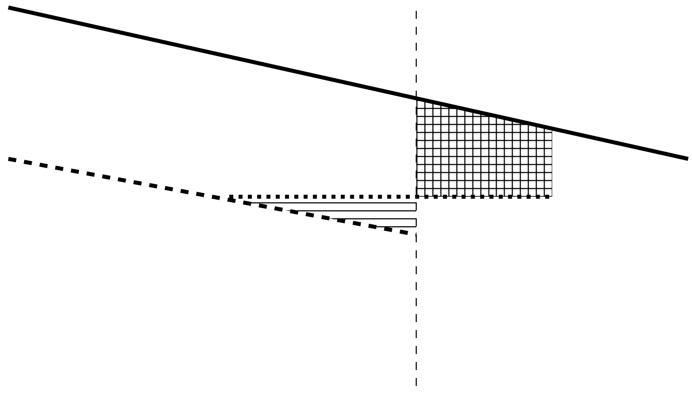

Figure 2: In equilibrium, bids must equal values at the maximum quantity which can be received (Theorem 1). Otherwise, a small upward deviation can obtain a discretely greater quantity (hashed area) at minimal additional cost (lined area).

measure-zero changes to bids have no effect on utility or incentives, without loss of generality in the sequel we assume that, for all bidders i,

$$b^i(\overline{Q}^R/n) = p(\overline{Q}^R) = v(\overline{Q}^R/n),$$

and we treat these quantities as deterministic. Furthermore, Theorem 1 allow us to speak unambiguously about the minimum market-clearing price (given the fixed signal s), $\underline{p} = v(\overline{Q}^R/n)$.[35] The equality of the market price at the maximum supply and each bidder's marginal value at the last unit they receive is illustrated in Figure 3.

The intuition for this theorem is that a bidder with a strictly positive margin at the maximum feasible quantity could slightly increase their bid and obtain a non-negligible additional quantity at minimally higher price, which would be a profitable deviation. Figure 2 illustrates this intuition and the proof of Theorem 1 (in Appendix D) formalizes it, taking care of technical complications related to mixed strategies, tie-breaking, flat bids, and binding monotonicity constraints. Of course, this intuition applies only to the maximum quantity at which the increased bid is paid only when it is marginal; at any lower quantity the increased bid would need to be paid also when inframarginal, hence bids will in general be below values for lower quantities.

Theorem 1 plays a crucial role in the equilibrium uniqueness result for symmetrically

[35] Theorem 1 determines the minimum market price because the market price is weakly decreasing in total quantity sold (an implication of bids being weakly decreasing in quantity), and hence the market price is minimized at effective maximum supply $\overline{Q}^R$. The market-clearing price at supply lower than $\overline{Q}^R$ can (and frequently does) rise above the lower bound $v(\overline{Q}^R/n)$.

informed bidders we state next, and therefore in many of our subsequent results.

4.2 Existence, Uniqueness, and Bid Representation

We first show that equilibrium is unique and tractable whenever it exists. The existence of equilibrium can then be analyzed in terms of what equilibrium strategies must be, if an equilibrium exists. We therefore defer discussion of existence until after our uniqueness and representation results, and for expositional simplicity our uniqueness and representation results are formulated conditional on the existence of Bayesian Nash equilibrium. Proofs of all results may be found in Appendix E.

We focus on *relevant quantities*, by which we mean the quantities that a bidder can win with positive probability in equilibrium. We say that an equilibrium is essentially unique if the set of relevant quantities and the bids on relevant quantities are the same in all equilibria; in particular, the market-clearing price, payments, and allocations conditional on the realization of supply is then the same in all equilibria; bids for quantities which the bidder never receives do not need to be uniquely determined.

Theorem 2. [Uniqueness] *The Bayesian Nash equilibrium is essentially unique.*

To get a sense why this theorem obtains, note that if we restricted attention to symmetric and smooth equilibria satisfying the first order condition (which we do not), then uniqueness would follow from Theorem 1. Indeed, in a symmetric smooth equilibrium bidders' first-order conditions give us an ordinary differential equation and Theorem 1 provides us with a unique initial condition for this equation by uniquely determining the price $p(\overline{Q}^R)$ at the maximum supply and hence, in a symmetric equilibrium, the bids for quantity $\overline{Q}^R/n$. The proof builds on this idea and addresses the difficulties raised by potential asymmetries, non-differentiabilities, and discontinuities.[36]

Our analysis of uniqueness allows us to construct equilibrium bidding strategies, which turn out to be surprisingly tractable. We formulate the strategies using the auxiliary concept of a *weighting distribution* (discussed after the theorem): for any quantity $Q \in [0, \overline{Q})$, the

[36]Our uniqueness result stands in contrast to nonuniqueness results in uniform-price auctions (cf. Klemperer and Meyer [1989]) and in Bertrand competition with convex costs (cf. Weibull [2006]). We discuss uniform-price auctions in Section 6. In Bertrand competition, convex costs correspond to our decreasing marginal value curve. We obtain uniqueness where Bertrand competition allows nonuniqueness because our bidders' strategy space is larger. In particular, Bertrand competitors who undercut must supply all market demand whether or not doing so is profitable, while our bidders may submit a limit bid which yields them only as much quantity as they desire. For a discussion of uniqueness in Bertrand competition see, e.g., Burguet and Sákovics [2017].

n-bidder weighting distribution has c.d.f.

$$F^{Q,n}(x) = 1 - \left(\frac{1 - F(x)}{1 - F(Q)}\right)^{\frac{n-1}{n}}.$$

Note that $F^{Q,n}$ has support $[Q, \overline{Q}]$ and increases from 0 when $x = Q$ to 1 when $x = \overline{Q}$; further discussion of $F^{Q,n}$ follows our bid representation result.

Theorem 3. [Bid Representation] *The essentially unique equilibrium is symmetric. For any quantity $q \in [0, \overline{Q}^R/n]$, the bid b^i of each bidder i is given by*

$$b^i(q) = \int_{nq}^{\overline{Q}} v\left(\frac{\min\left\{x, \overline{Q}^R\right\}}{n}\right) dF^{nq,n}(x). \tag{1}$$

We impose no assumptions on symmetry of equilibrium bids, their strict monotonicity, nor continuity or differentiability; we derive all these properties. Because the unique equilibrium is symmetric, the bid functions allow us to express the market price for any realization of supply $Q \in [0, \overline{Q}]$ as

$$p(Q) = b^i\left(\frac{Q}{n}\right) = \int_{Q}^{\overline{Q}} v\left(\frac{\min\left\{x, \overline{Q}^R\right\}}{n}\right) dF^{Q,n}(x). \tag{2}$$

Furthermore, when the reserve price does not bind, formulas (1) and (2) simplify, as $\overline{Q}^R = \overline{Q}$ and $\min\left\{x, \overline{Q}^R\right\} = x$; in this case the equilibrium bid equation can be rewritten as

$$b^i(q) = \int_{nq}^{\overline{Q}^R} v\left(\frac{x}{n}\right) dF^{nq,n}(x).$$

When the reserve price is binding, $R > v\left(\overline{Q}\right)$, the bid function is the same as if the supply was distributed on $\left[0, \overline{Q}^R\right]$ with a mass point at $\overline{Q}^R$.

The weighting distributions depend only the number of bidders and the distribution of supply, and not on any bidder's true demand. As the number of bidders increases the weighting distributions put more weight on lower quantities. In the limit, on its support $F^{Q,n}(x)$ converges to $\frac{F(x)-F(Q)}{1-F(Q)}$; that is, to the distribution of supply conditional on it being above Q. We can re-express the bid function in terms of per-capita supply as

$$b(q) = \int_{q}^{\overline{Q}^{\text{per capita}}} \max\left\{v(x), v\left(\overline{Q}^{R,\text{per capita}}\right)\right\} \frac{f^{\text{per capita}}(x)}{1 - F^{\text{per capita}}(q)} \left(\frac{n-1}{n}\right) \left(\frac{1 - F^{\text{per capita}}(q)}{1 - F^{\text{per capita}}(x)}\right)^{\frac{1}{n}} dx,$$

where $\overline{Q}^{\text{per capita}} = \overline{Q}/n$, $\overline{Q}^{R,\text{per capita}} = \overline{Q}^{R}/n$, $F^{\text{per capita}}(q) = F(nq)$ and $f^{\text{per capita}}$ is this c.d.f.'s density. When the number of bidders becomes large, holding per capita supply constant, the right-hand multiplicands approach $f^{\text{per capita}}(x)/(1-F^{\text{per capita}}(q))$, which is the conditional density at x given that realized per-capita supply is at least q; this limit is approached fairly rapidly as $\frac{n-1}{n}\left(\frac{1-F^{\mu}(q)}{1-F^{\mu}(x)}\right)^{\frac{1}{n}}$ approaches 1 rapidly (the impact on bids is depicted in Figure 5). Thus, in the limit, the theorem expresses the bid for quantity q as the average marginal value for the marginal unit, conditional on receiving quantity above q. In other words, in the large-n limit the bid on any relevant quantity q is equal to the expected Walrasian market clearing price conditional on the bidder receiving q, which is the event when changing the bid for unit q might affect the bidder's ex post payoff; a corresponding limit economy result is established in Swinkels [2001]. In the competitive limit the bidder bids away all marginal rents. Expected utility is still positive since marginal utility is decreasing in quantity, hence bidding away marginal rents leaves rents for inframarginal units.

Away from the competitive limit, the bidder might retain rents not only on inframarginal units but also on marginal units. The fewer bidders are in the auction, the more market power the bidders have and the higher are their rents on marginal units: this is reflected in the exponent $(n-1)/n$ in the weighting distribution $F^{Q,n}$. The equilibrium bids b^i are appropriately-weighted averages of bidders' marginal values v, and in this they resemble both the bids in the competitive limit and the bids in first-price auctions with privately-informed bidders. Because marginal values are decreasing in quantity, bids are below values—that is, bidders are shading their bids—except for the bid on the effective maximum quantity where limit equality obtains, an equality consistent with Theorem 1.[37]

In the special case when supply is deterministic, our bid representation implies that the bid function is flat on quantities up to $\overline{Q}^{R}/n$. It can be easily seen that flat bids can be supported in an equilibrium. Given deterministic supply the bidders know exactly the quantities they will receive in equilibrium: a deviation increasing the bid for lower quantities increases the payment to the seller without improving the bidder's allocation; a deviation decreasing the bid decreases the allocation and the decrease discourages the deviation provided opponents' bids on quantities above $\overline{Q}^{R}/n$ are sufficiently high.

As an example note that when marginal values v are linear and the supply distribution F is generalized Pareto, $F(x) = 1-\left(1-\frac{x}{\overline{Q}}\right)^{\alpha}$ for some $\alpha > 0$, then our bid representation shows that the equilibrium bids are linear in quantity. The linear-Pareto case of our general setting has been analyzed by Ewerhart et al. [2010] and Ausubel et al. [2014], who constructed the linear equilibrium directly in terms of the slope and incident of demand and the parameters of the Pareto distribution. Our general results contribute to our understanding of this example

[37] Wittwer [2018] discusses further intuition behind our representation.

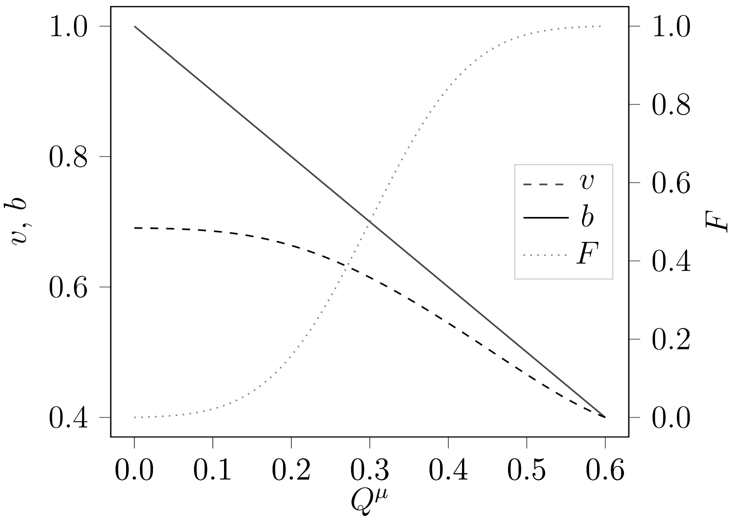

Figure 3: Equilibrium bids when values are linear and the distribution of supply Q is truncated normal. This and the subsequent figures represent bids, marginal values, and the c.d.f. of supply; to easily identify the three curves note that bids and the marginal values are decreasing, bids are below marginal values, and the c.d.f. is increasing.

by allowing us to conclude that the linear equilibrium is essentially unique in the class of all pure-strategy equilibria, and that bids remain linear in the linear-Pareto setting even in the presence of a reserve price. Figure 3 illustrates equilibrium bids in a distinct example in which ten bidders with linear marginal values face a distribution of supply that is truncated normal.[38]

Our bid representation theorem allows us to establish when an equilibrium exists because it derives the unique equilibrium bids on relevant quantities, conditional on equilibrium existence. When these bids are played in an equilibrium, we can express the expected utility of a bidder i as

$$\mathbb{E}[u^i] = \int_0^{\overline{Q}^R/n} U\left(q;q\right) dq,$$

where $U : [0, \overline{Q}^R/n]^2 \to \mathbb{R}$ is given by

$$U\left(\hat{q};q\right) = \left(v\left(q\right) - b\left(\hat{q}\right)\right)\left(1 - F\left(q + \left(n-1\right)\hat{q}\right)\right),$$

and b is the bid function derived in Theorem 3.[39]

Theorem 4. [Existence] *There exists a pure-strategy Bayesian Nash equilibrium in the*

[38]In all figures, we check our equilibrium existence condition and draw bids numerically using R. In Figure 3 we use a normal distribution with mean 3 and standard deviation 1, truncated to the interval $[0, 6]$.

[39]This expression for utility can be obtained via integration by parts; see footnote 63.

pay-as-bid auction whenever, for almost every $q \in [0, \overline{Q}/n]$, the first derivative of $U(\cdot; q)$ is zero only at the global maxima of $U(\cdot; q)$.

The proof of this theorem extends the bidding strategies $b^j(q) = b(q)$ from Theorem 3 beyond relevant quantities q and shows that then $\int_0^{\overline{Q}^R/n} \max_{\hat{q} \in [0, \overline{Q}^R/n]} U(\hat{q}; q) dq$ is an upper bound on the bidder i's expected utility for any bidding strategy. This approach allows us to verify pointwise that b is a best response.

Our sufficient condition is satisfied when, for example, the function $U(\cdot; q)$ is pseudo-concave, and hence also when $U(\cdot; q)$ is concave. The condition is also satisfied when the distribution of supply is deterministic. Additionally, our sufficient condition is closed with respect to several changes of the environment: adding a bidder, making marginal values less concave (or more convex), and raising the reserve price all preserve existence. In regular problems, the existence condition is satisfied as soon as there sufficiently many bidders.

Corollary 1. [Existence with many bidders] *Suppose marginal values are differentiable and have slope bounded away from zero, and the density of per-capita supply is bounded away from* 0 *on* $\left(0, \overline{Q}\right)$ *and has bounded derivative. If there are sufficiently many bidders, then a pure-strategy Bayesian Nash equilibrium exists.*

Regardless of market size, our sufficient condition is satisfied in the aforementioned linear-Pareto environment and it is satisfied whenever the inverse hazard rate H is increasing—hence when the hazard rate is decreasing—irrespective of the marginal value function v.[40] Our existence condition is satisfied in the examples illustrated in Figures 3-5, which include a truncated normal distribution, strictly concave marginal values, and reserve prices.

Our existence condition is also satisfied when supply is deterministic. Suppose that the seller commits to supply quantity $\overline{Q}$. As supply is deterministic, the auxiliary density $dF^{Q,n}(x)$ is equal to 0 for all $x < \overline{Q}$, and equilibrium bids are flat; the expression $(v(q) - b(\hat{q}))(1 - F(q + (n-1)\hat{q}) = U(\hat{q}; q)$ is therefore constant on $\hat{q} \in [0, \overline{Q}^R/n]$. Recall that we independently verified the existence of equilibrium in the deterministic case in our discussion of Theorem 3.

While our sufficient condition shows that equilibrium exists in many cases of interest, there are situations in which the equilibrium does not exist; see our discussion in the Introduction.

[40] The existence of equilibrium in the linear-Pareto environment was established by Ewerhart et al. [2010] and Ausubel et al. [2014] for bounded generalized Pareto distributions and Wang and Zender [2002], Federico and Rahman [2003], and Holmberg [2009] for unbounded Pareto distributions. The sufficiency of decreasing hazard rate for equilibrium existence was established by Holmberg [2009]. Theorem 4 also implies the existence results of Jackson and Swinkels [2005] and of Jackson and Kremer [2006], who showed that an equilibrium exists in the limit as per-capita supply goes to zero.

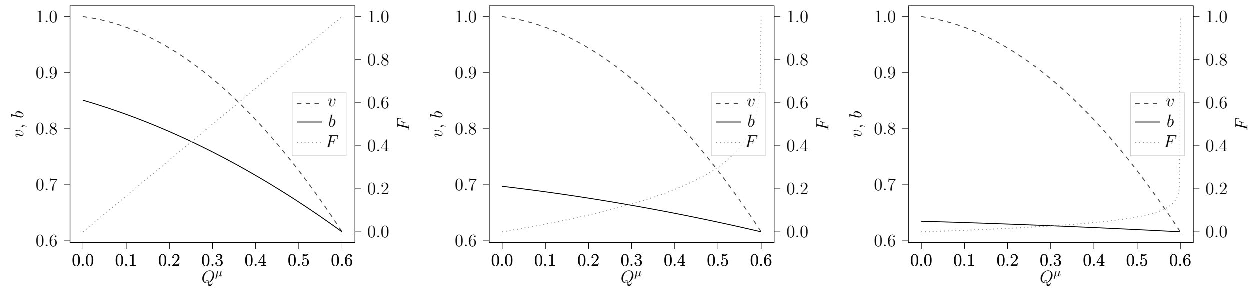

Figure 4: Bids are flatter for more concentrated distributions of supply.

4.3 Comparative Statics

Our bid representation implies that supply concentration leads to flat bids and low margins on bids near the per-capita concentrated quantity. We say that a distribution is δ-concentrated near quantity Q^* if $1-\delta$ of the mass of supply is within δ of quantity Q^*.

Corollary 2. **[Flat Bids]** *For any $\varepsilon > 0$ and quantity Q^* there exists $\delta > 0$ such that, if supply is δ-concentrated near $Q^* \leq \overline{Q}^R$, then the equilibrium bids for all quantities lower than $\frac{Q^*}{n} - \varepsilon$ are within ε of $v\left(\frac{Q^*}{n}\right)$.*

Bid concentration is especially straightforward to see in large markets, where bidders can affect their allocation but not the market-clearing price. In a large market each bidder picks the price they are willing to pay for each quantity, net of the unwillingness to overpay for this quantity when it is inframarginal. When per capita supply is concentrated at $Q^{\mu *}$, there is at worst a small probability that quantity $Q^{\mu *}$ will be inframarginal, hence the bidder is willing to pay nearly $v(Q^{\mu *})$.

Figure 4 depicts the flattening of equilibrium bids predicted by Corollary 2, in a moderately-sized market; in the three sub-figures ten bidders face supply distributions that are increasingly concentrated around the total supply of 6 (per capita supply of 0.6). In the special case of deterministic supply, which is 0-concentrated, Corollary 2 implies that equilibrium bids are perfectly flat.

The practical implications of Corollary 2 may be observed in U.S. Treasury auctions for short-term securities. Hortaçsu et al. [2018] show that in these auctions supply randomness is low, and empirically-observed uniform-price bids are nearly flat. Because supply randomness is low, Corollary 2 implies that counterfactual pay-as-bid bids would also be nearly flat, and changing the auction format would yield little additional revenue.[41]

[41] Hortaçsu et al. [2018] use inferred marginal values to show that bidders do not obtain much surplus; thus changing the auction format cannot yield much additional revenue. Our corollary goes beyond their analysis by showing that given flat uniform-price bids and relatively certain supply, changing the auction

Theorem 4 implies that if equilibrium exists in two component markets, then it exists in the merged market. Our bid representation further implies that bidders' equilibrium margins are lower and the seller's revenue is higher when there are more bidders:

Corollary 3. [More Bidders and Marketplace Mergers] *Bidders submit higher bids, the seller's revenue is higher, and each bidder's profits smaller when there are more bidders—both when the supply distribution is held constant, and when the per-capita supply distribution is held constant. In particular, the sum of revenues from markets with n_1 and n_2 bidders and the same per-capita supply distribution is less than the revenue from the joint market with $n_1 + n_2$ bidders.*

The corollary follows because as the number of bidders increases, $1-F^{Q,n}(x) = \left(\frac{1-F(x)}{1-F(Q)}\right)^{\frac{n-1}{n}}$ decreases, and hence $F^{Q,n}(x)$ increases, thus mass in the weighting distribution is shifted towards lower x, where marginal values are higher. At the same time, the marginal value at x either increases in n (if we keep the distribution of supply constant) or stays constant (if we keep the distribution of per-capita supply constant). Our bid representation also implies that when within-market per capita supply is similar across divided markets, merging the markets will improve total revenue; however, if the two markets have substantially different per capita supply, then merging them might decrease total revenue. Similar market-merger conclusions have been derived for uniform-price auctions, cf., e.g., Rostek and Yoon [2021], Fabra and Llobet [2021], Wittwer [2021]. On the other hand, Theorem 5 below implies that with optimal supply in both markets (and both markets having at least two bidders each), merging the markets will have no effect on revenue if the per-capita supply is the same in the markets being merged; if the per-capita supply differs across these markets than the merger increases the revenue if bidders' true marginal demands are concave but decreases the revenue if the true marginal demands are convex.

While bidders raise their bids when facing more bidders even if the per-capita distribution stays constant, our bid representation theorem implies that the changes are small; the intuitive reason is that as the number of bidders goes to infinity, our equilibrium construction converges to that in the large-market analysis of [Swinkels, 2001].[42] This is illustrated in Figure 5 in which increasing the number of bidders from 5 bidders to 10 bidders has only a small impact on the bids, as does the further increase from 10 bidders to 5 million bidders.

format also cannot cost much revenue. See Section 7 for further discussion of flat bids.

[42] If we keep the supply distribution fixed while more and more bidders participate in the auction, then in the large market limit revenue converges to average supply times the value on the initial unit. See Swinkels [2001] for limit results with fixed per-capita supply and Jackson and Kremer [2006] for limit results with fixed supply.

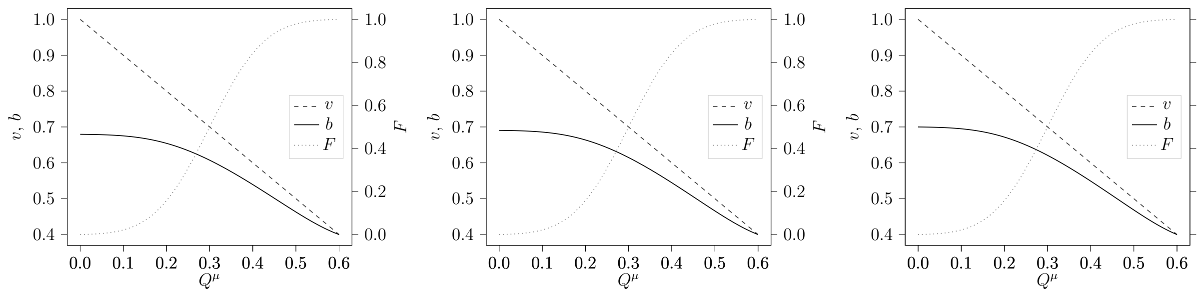

Figure 5: Bids go up when more bidders arrive (and per capita quantity is kept constant) but not by much: 5 bidders on the left, 10 bidders in the middle, and 5 million bidders on the right. Note that all axis scales are identical.

5 Designing Pay-as-Bid Auctions: Transparency and Disclosure

In this section we maintain the assumption that the pay-as-bid format is run and analyze the design of such auctions. We focus on the reserve price and the distribution of supply, the two natural elements of pay-as-bid auction that the seller can select, and we continue to impose the assumptions applied in the equilibrium analysis of Section 4.2; in particular we restrict attention to pure-strategy equilibria.[43] In Appendix A, we relax all these assumptions while also allowing elastic supply and mixed-strategy equilibria, and show that our results on transparency (Theorem 5) and disclosure (Theorem 6) remain valid.

As design decisions are taken from the seller's perspective, our terminology in this and the subsequent sections now explicitly keeps track of the bidders' information s.

5.1 Transparency

The key insight that underlies our design analysis is that—in contrast to typical multidimensional mechanism design problems discussed in the introduction—in an optimized pay-as-bid auction deterministic—and, hence, transparent—supply is optimal. Furthermore, if supply is exogenously random, then it is optimal for the seller set a deterministic supply cap; and, independent of whether a supply cap is feasible, it is optimal to announce the realized supply to the bidders prior to the auction.

[43]As pay-as-bid is largely employed by central banks and governments, the efficiency of allocations may be an important concern and a reason a seller may want to ensure that a equilibrium in pure strategies is being played. The symmetry of equilibrium strategies we prove in Theorem 3 implies that in such equilibria the marginal value for any unit received is higher than the marginal value for any unit not received. There are thus no efficiency improving re-allocations of units among bidders; this property trivially fails in any mixed-strategy equilibrium that is not essentially in pure strategies.

First, suppose that the seller has some deterministic quantity $\overline{Q}$ of the good; we relax this assumption below. For any fixed reserve price, we consider the problem of designing a supply distribution F that maximizes the seller's revenue. The seller has the option to offer a stochastic distribution over multiple quantities, up to $\overline{Q}$. In a treasury auction, a seller may commit to random supply sold at auction by setting it equal to a total supply net of sales to non-competitive buyers, a common practice in the treasury auctions in the U.S. [TreasuryDirect, 2022] and Japan [Hattori and Takahashi, 2022]. It is also plausible that such randomization could increase the seller's expected revenue. For instance, stochastically offering quantities lower than the optimal monopoly quantity $Q^{\star}$ (subject to the supply constraint), results in a tradeoff: the seller sometimes sells less than $Q^{\star}$, with a direct and negative revenue impact, but when he sells quantity close to $Q^{\star}$ or higher he may receive higher payments due to the pay-as-bid nature of the auction. This tradeoff is illustrated in Figure 4, in which concentrating supply lowers the bids.

We show that selling the deterministic supply $Q^{\star}$ is in fact revenue-maximizing; for this reason in the sequel we refer to $Q^{\star}$ as optimal supply.

Theorem 5. **[Transparency of Optimal Supply]** *The seller's revenue under non-deterministic supply is strictly lower than under optimal deterministic supply. Optimal deterministic supply is given by the solution to the monopolist's problem when facing uncertain demand.*

As the following proof sketch indicates, Theorem 5 remains valid if the reserve price is arbitrary rather than optimized. The theorem also remains valid for sellers who maximize profits equal to revenue net of costs, provided the marginal cost curve is weakly increasing. Such sellers optimally choose the deterministic quantity that maximizes the expected revenue minus cost rather than the quantity that maximizes the expected revenue. Taking the cost into account affects what quantity is optimal, but it does not change the result that optimal supply is deterministic.

To prove Theorem 5, we start with an arbitrary reserve price and supply distribution and the induced pure-strategy equilibrium bids. Holding equilibrium bids fixed, we use our bid representation from Theorem 3 to bound expected revenue by the standard monopoly revenue given the supply distribution.[44] In effect we obtain the following bound on the expected revenue,

$$\mathbb{E}_{s,Q}\left[\pi^{F}\left(Q;s\right)\right] \leq \int_{0}^{\overline{Q}^{R}} \mathbb{E}_{s}\left[\pi^{\delta_{Q}}\left(Q;s\right)\right] dF\left(Q\right), \tag{3}$$

[44] This argument hinges on re-assigning the revenue across supply realizations; in particular, the actual revenue conditional on a supply realization is not necessarily bounded by the revenue the seller would obtain by setting the deterministic supply fixed at the conditioning supply realization.

where $\pi^F(Q; s)$ is the seller's revenue when the bidders' signal is s, the realization of supply is Q, and bidders bid against the distribution of supply F, while $\pi^{\delta_Q}(Q; s)$ is the seller's revenue when the bidders' signal is s, the realization of supply is Q, and bidders bid against the supply distribution δ_Q that puts probability 1 on quantity Q. Note that $\pi^{\delta_Q}(Q; s)$ is a monopolist's profit from selling quantity Q to buyers with common signal s. This upper bound implies that the seller's revenue is maximized when the seller sets the supply to be always equal to the revenue-maximizing deterministic supply. We provide the details of the proof in Appendix G (bound (3) above restates inequality (13) in the proof).

The structure of the proof of Theorem 5 has two important implications. First, under the additional restriction that $Q\mathbb{E}_s[v^{-1}(Q; s)]$ is single-peaked in Q, the proof is applicable to environments in which the seller's underlying supply is random and the seller can lower the supply but cannot increase it above the underlying supply realization. In this more general environment we assume that the distribution of underlying supply is exogenously given by F with a compact support.[45] Our proof then shows that the revenue-maximizing supply reduction by the seller reduces supply to $Q^\star$ whenever the exogenous supply is higher than $Q^\star$, and otherwise leaves the supply unchanged. As discussed following Theorem 4 and Corollary 2, when supply is deterministic bids are flat at level $v(\overline{Q}/n; s)$.

In Appendix B we extend the transparency theorem to auctioneers whose revenue is the sum of revenue from the auction (accepted bids of competitive bidders) and revenue from noncompetitive demand filled at the price determined in the auction.

5.2 Full Disclosure

As an application of our analysis let us note that the seller who runs an auction with random supply would like to fully reveal the realized supply. For instance, in the United States [TreasuryDirect, 2022] and Japan [Hattori and Takahashi, 2022], the seller announces joint supply of debt to be sold in an auction and allocated to noncompetitive bidders, and the supply sold in an auction is then the residual supply after noncompetitive bidders' demand is filled. The seller thus finds transparency optimal both in the sense of setting a deterministic supply (or supply cap) and in the sense of revealing the seller's information about supply.

To formalize this full-disclosure insight we enrich our base model as follows. We assume that the distribution of supply is exogenously given and commonly known. Before learning the realization of supply, the seller can publicly commit to an auction design (reserve price and supply restriction) and a disclosure policy; a disclosure policy maps the realization of supply to a distribution of public announcements (messages) from an arbitrary space of

[45]The compactness of F's support could be replaced by other assumptions that guarantee that the optimal solution exists, such as for instance that there is a finite $q > 0$ such that for all s, $v(q; s) = 0$.

messages. After publicly committing to a disclosure policy and an auction design, the seller learns the realization of supply and announces the message prescribed by the disclosure policy. Then, the bidders learn their value and bid in the auction.

Theorem 6. [**Optimality of Information Disclosure**] *The seller's expected revenue is maximized when the seller commits to fully reveal the realization of supply.*

Before presenting a surprisingly simple argument deriving this theorem from our preceding analysis, let us observe that Theorem 6 remains valid even if the seller does *not* optimize the reserve price and supply cap in the auction and these parameters of the auction are arbitrarily set, with no change in the proof. In addition, because we prove Theorem 6 for the environment in which the seller can commit to a disclosure strategy, the same full disclosure insights a fortiori holds true for environments where the seller cannot commit.

Proof. Suppose that the seller commits to a disclosure strategy and this strategy leads to a message that induces the bidders to believe that the (conditional) distribution of supply is $\hat{F}$ with upper bound of support $\hat{Q}$. The revenue bound (3) gives

$$\mathbb{E}\left[\pi^{\hat{F}}\left(Q;s\right)\right] \leq \int_0^{\hat{Q}^R} \mathbb{E}_s\left[\pi^{\delta_x}\left(x;s\right)\right] d\hat{F}\left(x\right),$$

and thus expected revenue is bounded above by the expected revenue obtained by the seller fully revealing to the bidders the realization of supply. In consequence, the seller's expected revenue is maximized when the seller ex ante commits to fully reveal the realization of supply. □

As with Theorem 5, an analogue of Theorem 6 remains valid when the seller obtains revenue from noncompetitive demand, see Appendix B. Furthermore, in the supplementary note [Pycia and Woodward, 2023a], we show that the revenue-maximizing seller not only would like to reveal supply information but, if the seller has information relevant for bidders' valuations, the seller would like to release it as well.[46]

5.3 The Contrast with Uniform Price

The optimality of transparency and full revelation hinges on using the pay-as-bid format. In uniform-price auctions, it can be optimal to randomize supply and not disclose the realization of randomness; Section 6 defines these auctions and shows that a wide range of supply

[46] The New York Federal Reserve asked one of us (Pycia) what supply and demand related information to disclose to the bidders in US liquidity auctions. The NY Fed practice was to disclose as little as possible, while Mark Carney of the Bank of England advocated for transparent disclosure. Our results support the disclosure.

randomizations might be optimal. One reason to use randomization is to prevent a form of tacit collusion that has been observed in uniform price auctions. For instance, Harbord and Pagnozzi [2014] discuss the revelation of demand information in uniform-price procurement auctions for power generation capacity in Colombia and in New England, and Schwenen [2015] discusses uniform-price procurement for power capacity in New York; these papers show that the price in these auctions can be determined by tacit collusion, where submitted bids are too low to be profitably undercut on the margin (these papers study procurement auctions, in which bidding low corresponds to bidding high in our model). Increasing the randomness of supply could benefit the seller by breaking this equilibrium. Analogous equilibria do not occur in pay-as-bid, because the fringe bidders would need to pay their high bids (or sell at the low bids).

6 The Auction Design Game: Pay-as-Bid Dominates Uniform-Price

Sellers of homogeneous goods are not constrained to use pay-as-bid auctions. As we discuss in the Introduction, sellers usually choose between implementing a pay-as-bid auction or implementing a uniform-price auction, and which of these two formats is preferred remains an important open question. Earlier comparisons of these formats did not take the seller's endogenous choices into account. In this section we explicitly model the seller's choice between pay-as-bid and uniform-price formats, as well as among supply distributions and reserve prices, as an extensive-form game.

As in Section 5 we focus on the reserve price and the distribution of supply and we continue to impose the assumptions applied in the equilibrium analysis of Section 4.2; in particular we restrict attention to pure-strategy equilibria. In Appendix A, we show that our revenue comparisons (Theorem 7 and Corollary 5) remain valid after we relax all these assumptions and allow any random elastic supply and any mixed-strategy equilibria.

This auction design game has two stages. In the first stage, the seller commits to a reserve price, a distribution of supply, and the auction format (pay-as-bid or uniform-price). We also consider constrained design games in which the auction format is fixed; we refer to these as pay-as-bid design game and uniform-price design game. In the second stage, bidders participate in the specified auction.[47] We consider perfect Bayesian equilibria of

[47] The bid functions $b^i(\cdot; s, R, F)$ depend on the bidders' signal as well as the auction format and the reserve prices R and supply distributions F chosen by the seller. When there is no risk of confusion, when referring to the bids on the equilibrium path we sometimes suppress the seller's choices. Explicitly modeling both stages of the design game allows us to study the feedback loop between mechanism design and equilibrium

these games. This structure allows us to compare outcomes of optimally designed pay-as-bid and uniform-price auctions, and to discuss the economic implications of mechanism selection. Our main insight is that choosing pay-as-bid is weakly dominant for the seller.

6.1 Uniform-Price Auctions

As discussed above, uniform-price auctions are the main alternative to the pay-as-bid auction format. In the uniform-price auction, the space of feasible bids, the market-clearing price $p^\star$, and allocations q_i are defined in the same way as in pay-as-bid (see Section 3). The only feature distinguishing the two formats is the bidders' payment rule: instead of paying their own bids, in the uniform-price format each bidder i pays a constant market price per unit, hence bidder i's payment is $p^\star q_i$.

As mentioned in Section 5.3, in a uniform-price auction it may be optimal to commit to random supply. A key reason this might happen is the failure of equilibrium uniqueness in uniform price. Because bidders' continuation equilibrium can be selected based on the chosen distribution of supply, it is possible that choosing deterministic supply will yield lower revenue than random supply: when bidders play a low-revenue equilibrium when supply is deterministic (or close to deterministic), and play a high-revenue equilibrium otherwise, the seller may optimally concentrate the supply distribution around the deterministic optimum while retaining some randomness to ensure that bidders submit aggressive bids. The construction of such equilibria relies on the value space being *rich* in the following sense: the set $\{s\colon v(Q^\star/n;s) > R^\star\}$ has positive probability for all deterministic supply and reserve pairs $(Q^\star, R^\star)$ that maximize monopoly revenue,

$$\begin{aligned}(Q^\star, R^\star) \in \arg\max_{Q,R}\ & R\mathbb{E}\left[nv^{-1}(R;s)\middle| v(Q/n;s) < R\right]\Pr\left(v(Q/n;s) < R\right) \\ & + Q\mathbb{E}\left[v(Q/n;s)| v(Q/n;s) \geq R\right]\Pr\left(v(Q/n;s) \geq R\right). \end{aligned} \tag{4}$$

Richness rules out the complete information case, which we discuss separately in Corollary 6.

Lemma 1. [Quantity and Reserve in Uniform Price] *Suppose the value space is rich*

selection. The presence of this feedback loop gives our analysis more predictive power than the standard focus on the bidding stage alone. For instance, the Perfect Bayesian Equilibria do not allow designs that lead to revenues lower than the max-min second-stage revenue, where the max is taken over designs and min over second-stage equilibria. The explicit modeling of both stages also allows us to formally state when revenue-maximizing sellers choose uniform price auction (Corollary 5). At the same time, the two-stage approach is equivalent to the focus on the bidding (second) stage when the auction has unique equilibrium (as in pay as bid) or when we restrict attention to a uniquely selected equilibrium (e.g. robust equilibria of Section 6.1).

and let $R^{\star PAB}$ and $Q^{\star PAB}$ be optimal reserve and supply in the pay-as-bid design game. There is $\varepsilon > 0$ such that for all reserve prices $R \in [R^{\star PAB} - \varepsilon, R^{\star PAB} + \varepsilon]$ and all supply distributions F with support in $[Q^{\star PAB} - \varepsilon, Q^{\star PAB} + \varepsilon]$, there is an equilibrium of the uniform-price design game in which the designer selects reserve R and supply distribution F.

The proof builds on the construction of two equilibria classes:

- Robust equilibrium, defined as a profile of strategies that is an equilibrium for all distributions of supply; the existence and uniqueness of such an equilibrium follows from Klemperer and Meyer [1989]; and
- Semi-truthful equilibria, defined as equilibria at which $b^{\text{UPA}}(\overline{Q}^R/n; s) = v(\overline{Q}^R/n; s)$.

Appendix G.1 constructs both these equilibria classes and shows that, under the richness assumption, the expected revenue from the robust equilibrium following any reserve and supply distribution is strictly lower than (and bounded away from) the expected revenue from a semi-truthful equilibrium following reserve $R^{\star\text{PAB}}$ and deterministic supply $Q^{\star\text{PAB}}$. The perfect Bayesian equilibrium implementing reserve R and supply distribution F is then constructed as follows. If the seller sets R and F then, in the continuation game, bidders play the constructed semi-truthful equilibrium. If the seller sets different reserve or different distribution of supply then, in the continuation game, the bidders play the robust equilibrium, which has comparatively low bids. As ε goes to 0, the expected revenue in the semi-truthful continuation equilibrium approximates that in the semi-truthful continuation equilibrium following reserve $R^{\star\text{PAB}}$ and supply $Q^{\star\text{PAB}}$. As the difference between the expected revenue in robust and semi-truthful equilibria following $R^{\star\text{PAB}}$ and $Q^{\star\text{PAB}}$ is bounded away from zero, for all R and F within sufficiently small ε of $R^{\star\text{PAB}}$ and $Q^{\star\text{PAB}}$ (respectively), the expected revenue from setting R and F is strictly higher than the revenue from any other reserve and supply distribution.

6.2 Revenue

For the pay-as-bid auction, Theorem 2 states that equilibrium bids are essentially unique conditional on the distribution of supply, and Theorem 5 states that optimal supply is deterministic. Together these theorems imply that equilibrium revenue is unique in the pay-as-bid design game.

Corollary 4. [Revenue in Pay-as-Bid Design Game] *In the pay-as-bid design game with symmetrically informed bidders, the perfect Bayesian equilibrium revenue is uniquely determined and the seller can achieve it by setting optimal deterministic supply.*

Revenue analysis of the uniform-price design game is more complicated: as we have seen in the previous subsection randomness might be optimal on the path of a particular equilibrium. Despite this we show in Lemma 16 in Appendix G that the maximum revenue in uniform-price design game is obtained in a perfect Bayesian equilibrium in which the seller sets the same reserve price and deterministic supply as in revenue-maximizing pay as bid. In consequence, any equilibrium of the uniform-price game generates weakly less revenue than the unique expected revenue in any equilibrium of the pay-as-bid design game.

Theorem 7. [**Revenue Comparison of Design Games**] *The expected revenue of the pay-as-bid design game is weakly greater than the expected revenue in any equilibrium of the uniform-price design game.*

The revenue comparison is strict for all uniform-price equilibria in which bidders are not semi-truthful. The non-semitruthful equilibria are typical in the sense that in the uniform-price auction, for any reserve R, supply distribution F, and signal s, the set of prices at maximum supply $\overline{Q}^R$ that are supportable in equilibrium is the interval $[R, v(\overline{Q}^R(s)/n; s)]$. In particular, robust equilibria are not semi-truthful and the ranking of pay as bid and uniform price becomes strict for robust equilibria. At the same time, there is a semi-truthful equilibrium of the uniform-price design game that generates the same expected revenue as the unique equilibrium revenue of the pay-as-bid design game. The theorem and these claims remain valid for any deterministic distribution of supply; for their proofs see Appendix G.

Theorem 7 implies that in the auction design game in which the designer chooses either a pay-as-bid or uniform-price format, and its reserve price and supply distribution, the seller will either implement a pay-as-bid auction or, expecting the bidders to bid semi-truthfully in uniform price, is indifferent between the two formats.

Corollary 5. [**Revenue Equivalence across Perfect Bayesian Equilibria**] *All perfect Bayesian equilibria of the auction design game are revenue equivalent. Furthermore, the seller either implements a pay-as-bid auction or is indifferent between the pay-as-bid and uniform-price auctions.*

Finally, when the seller has access to the bidders' information at the time the auction is designed, the optimal pay-as-bid auction is outcome-equivalent to simply posting the monopoly-optimal price. Because posting the monopoly-optimal price is also feasible in the uniform-price auction, it follows that when there is symmetric information between the buyers and the seller, the pay-as-bid and uniform-price formats are revenue equivalent when optimally designed.

Corollary 6. [Revenue Equivalence with an Informed Seller] *When the buyers' signal s is known to the seller, then the optimally designed uniform-price auction has a unique equilibrium, and this equilibrium is revenue-equivalent to the optimal pay-as-bid auction.*

7 Relationship to Empirical Findings

As discussed in the Introduction, an extensive empirical literature studies the use of the pay-as-bid and uniform-price auctions in real-world settings. Our model and main results correspond to empirical features observed across these studies. First, while empirical work provides no clear guidance on which of the pay-as-bid or uniform-price auction formats raises greater expected revenue in general, Table 2 shows that, across studies where supply randomness is reported, pay-as-bid dominates when supply randomness is small. This observation is consistent with our transparency result (Theorem 5), which shows that when supply is deterministic the pay-as-bid auction raises strictly greater revenue than all but the seller-optimal equilibrium of the uniform-price auction (a result whose robustness to the presence of asymmetric information we verify in the supplementary note Pycia and Woodward [2023a]).

An important prediction of our model is that bids are approximately flat when outcomes are relatively certain (Corollary 2); conversely, when outcomes are relatively uncertain bidders will hedge against low allocations by bidding more aggressively for low quantities. Given a bidder's uncertainty, flatness is a property of best-responses and does not hinge on the bids being in equilibrium. We can use this prediction to test the validity of the assumption that bidders are approximately symmetrically informed. Bid flatness has been observed in empirical analyses of European liquidity pay-as-bid auctions prior to the crisis of 2007 [Cassola, Hortaçsu, and Kastl, 2013], as well as Canadian [Hortaçsu and Sareen, 2005], South Korean [Kang and Puller, 2008], Chinese (Barbosa et al., 2020, and Yoshimoto, 2021, private communication), and Polish (Marszalec, 2017, and Marszalec, 2021, private communication) pay-as-bid treasury auctions, indicating that bidders face little relevant asymmetric information or other uncertainty in these auctions.[48] Another natural test of the symmetry assumption is the difference between auction price and the subsequent secondary market price; this difference is small in auctions for which we found data (on average 0.04% of the clearing price in Finnish auctions studied by Keloharju et al. [2005], and on average 0.09%

[48]The New York Times [1929] reports that flat bids were observed in pay-as-bid U.S. Treasury auctions as early as the 1920s. The yield tail—the difference between the average accepted yield and the market-clearing yield—in U.K. Conventional Gilt auctions between March 2021 and March 2023 was 1.14bp (own calculation), consistent with relevant bids being flat. In addition, Hortaçsu et al. [2018] observe flat bids in uniform-price U.S. Treasury auctions.

Paper	**Data**	**Method**	σ/μ	**# Bidders**	**Conclusion**
Marszalec [2017]	Poland	PAB $\to$ CF UP	0.00%	12.3	PAB > UP
Barbosa et al. [2020]	China	Controlled exp.	0.00%	35.2	PAB $\approx$ UP
Février et al. [2002]	France	PAB $\to$ CF UP	1.27%	20.8	PAB > UP
Armantier and Sbaï [2006]	France	PAB $\to$ CF UP	3.78%	19.0	UP > PAB
Hattori and Takahashi [2022]	Japan	Natural exp.	11.00%	no data	PAB > UP
Umlauf [1993]	Mexico	Natural exp.	11.16%	24.7	UP > PAB
Mariño and Marszalec [2020]	Philippines	Natural exp.	17.60%	20.3	PAB > UP

Table 2: Revenue comparisons between auction formats, in comparison to the standard deviation of noncompetitive demand scaled by mean aggregate supply (Q); "CF" is "counterfactual."

of the clearing price in U.K. Conventional Gilt auctions (own calculation)).[49]

Our results on the auction design game suggest that the auctioneer's choice of auction format will carry information about which auction format yields greater revenue. Cross-country comparisons find that both pay-as-bid and uniform-price auctions are popular (see OECD [2021] and Brenner et al. [2009] for treasury securities, Maurer and Barroso [2011] for electicity generation, and Del Río [2017] for electricity generation capacity), and our results provide a theoretical explanation for the popularity of the pay-as-bid format. Our Theorem 3 and Proposition 1 imply that, in large competitive markets, pay as bid and robust bids in uniform price will raise similar revenue, while in smaller markets pay as bid is likely to be revenue dominant.[50] Of course, our predictions are only a baseline, and the auctioneer may be interested in outcomes beyond revenue.

Finally, Corollary 5 provides an explanation of the empirical finding that revenues in pay-as-bid are close to the counterfactual revenues in uniform price, as discussed in the Introduction. The explanation is twofold. First, the Corollary shows that a revenue-maximizing seller weakly prefers the uniform-price format only if this format is equivalent to pay as bid. The South Korean Treasury auctions studied by Kang and Puller [2008] and U.S. Treasury auctions studied by Hortaçsu, Kastl, and Zhang [2018] run the uniform-price format

[49] We apply Keloharju et al.'s methodology to U.K. Conventional Gilt sales between March 2021 and March 2023 for which the market-clearing price is available. Note that, unlike the gap between primary- and secondary-market prices, even a slight amount of asymmetric information might induce significant asymmetries in bidders' ex post allocations. Hence the presence of such asymmetries would not falsify the (nearly) symmetric information assumption. As we prove in our companion note Pycia and Woodward [2023a], such asymmetries have no substantive impact on revenue or the choice of revenue-maximizing mechanism; in particular, the approximate analogue of revenue-equivalence Corollary 5 continues to hold.

[50] In our supplementary note [Pycia and Woodward, 2023a] we provide a large market revenue equivalence result, consistent with earlier large market results, cf. Swinkels [2001]. Our bid representations go further by making explicit the dependence of bids on the number of bidders (cf. Corollary 3).

and hence Corollary 5 provides a potential explanation of the revenue equivalence found in these papers. Second, the optimal pay-as-bid and uniform-price auctions generate the same revenue only in the seller-optimal equilibrium of the uniform-price auction and this is precisely the equilibrium in which bids are equal to marginal values at realized quantities. The latter equality is imposed in counterfactual revenue estimation of uniform-price auctions in Hortaçsu and McAdams [2010] and Marszalec [2017] which assume truthful bidding in the uniform-price auction; as these papers discuss, the imposed assumption results in an upper bound on uniform-price revenue. The counterfactual assumption of truthful bidding in the uniform-price auction is likely to bias expected revenues upwards when supply randomness is high as implied by the equilibrium analysis of Klemperer and Meyer [1989]; when supply randomness is low pay as bid is approximately revenue equivalent to truthful bidding in uniform price. The theory thus suggests that the empirical ambiguity of cross-mechanism revenue comparison might be tied to sellers' endogenous selection of auction format and to the counterfactual strategy selection in the empirical literature.

8 Conclusion

We have studied multi-unit auctions in an environment in which bidders have symmetric information, but the seller (or auction designer) is potentially much less informed. We have established a mild condition for equilibrium existence as well as established equilibrium uniqueness and provided a tractable representation of bids. We hope that the tractability of our representation will stimulate future work on this important auction format.

We have used these results to analyze the design problem of the seller. In particular, we established that revenue-maximizing pay-as-bid auctions generate more revenue than uniform-price auctions, and strictly more revenue in most cases; welfare comparisons are inherently ambiguous. In particular, it is possible that revenue-maximizing pay-as-bid auctions are not only revenue- but also welfare- superior to uniform-price auctions.

As part of our analysis we established revenue equivalence between revenue-maximizing pay-as-bid auctions and the revenue-maximizing equilibrium of uniform-price auctions. Our revenue equivalence benchmark—which we prove both for optimally-designed auctions and for deterministic supply—provides an explanation for the empirical findings of approximate revenue equivalence between the two formats.

In our supplementary note Pycia and Woodward [2023a], we show that all our design results are robust to the presence of small informational asymmetries among bidders. An analogue of Theorem 1 continues to hold in asymmetric information environments, and we use it to bound the revenue differences in the pay-as-bid auction between symmetric-information and asymmetric-information environments: analogously to Theorem 7 we show "approximate revenue dominance" of pay-as-bid over uniform-price in environments with asymmetric information; analogously to Corollary 5 we show that a revenue-maximizing seller would select uniform price only if expecting it to be approximately revenue-equivalent to pay as bid. The basic insight that pay-as-bid equilibria in asymmetric information environments converge to the symmetric information equilibria as the asymmetric information shrinks is a corollary of Reny [1999].[51] In our supplementary note [Pycia and Woodward, 2023a] we also show that our design results remain valid for sellers with increasing marginal costs; we study the relationship of the pay-as-bid auctioneer to a classical monopolist and discuss the connection between the present analysis and dynamic oligopoly; we show that revealing information on bidders' values increases seller's revenue; and we show that the auctioneer's design problem is separable, and that the decisions of optimal supply and optimal reserve may be analyzed independently.

In follow-up work [Pycia and Woodward, 2023b] we analyze the problem of efficient allocation of permits in emissions markets. The dominance of pay-as-bid over uniform-price, which we establish in a revenue-maximization context in this paper, holds with respect to surplus maximization as well. Key to this analysis is an extension of Theorem 1 to settings where bidders may be ex ante and interim asymmetric. Taken together, our work shows that the pay-as-bid auction format may have several underappreciated advantages over the uniform-price auction format.

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A Elastic Supply

In the main text we (mostly) focus on pure strategy-equilibria and on designing a potentially stochastic supply distribution allowing for a separately set reserve price. Our essential insights remain valid if we allow mixed-strategy equilibria and potentially stochastic elastic supply curves.

We study a seller who selects a distribution over reserve prices, possibly correlated with the distribution of quantity. Let $K(Q;R)$ be a *supply-reserve distribution*, giving the probability that realized quantity is $\tilde{Q} \leq Q$ or the realized reserve price is $\tilde{R} > R$,

$$K\left(Q;R\right) = \Pr\left(\tilde{Q} \leq Q\right) + \Pr\left(\tilde{Q} > Q, \tilde{R} > R\right).$$

Conditional on aggregate demand $p(\cdot)$, $K(Q;p(Q))$ is the probability that realized aggregate supply is below Q: either realized supply is $\tilde{Q} \leq Q$, or realized reserve is $\tilde{R} > p(Q)$ and quantity is constrained. While K is not a c.d.f., it describes the joint distribution over quantity and reserve. The following special cases illustrate the supply-reserve distribution K:

- K is equivalent to a random supply distribution F if $K(Q,R) = F(Q)$;
- K is equivalent to a random reserve distribution F^R if $K(Q,R) = 1 - F^R(R)$;
- K is equivalent to deterministic supply curve S if $K(Q,R) = 1[S(R) < Q]$.

To key to extending our results to this environment is establishing the analogues of our uniqueness and transparency results. Equilibrium uniqueness obtains when the elastic supply curve is deterministic because an analogue of Theorem 1 obtains (see Appendix H for details of this and other proofs).

Theorem 8. [Unique Pay-as-Bid Equilibrium] *If the elastic supply is deterministic then the pay-as-bid auction admits an essentially unique equilibrium.*

In the essentially unique equilibrium, all bidders bid their marginal value on the last allocated unit for all units they receive; they can randomize over their bids on units the do not receive with no impact on equilibrium outcome.[52]

Perhaps paradoxically, the main difficulty in proving the optimality of deterministic elastic supply lies in establishing this result for the case when the bidders' common signal, s, is known to the seller—that is when it takes a constant value with probability 1.

Lemma 2. [Deterministic Dominance when the Seller Knows Bidders' Signal] *Suppose bidders' information is known to the seller. Given any supply-reserve distribution K, there is a deterministic quantity $Q^\star$ such that the pay-as-bid auction with fixed supply $Q^\star$ raises greater revenue than the pay-as-bid auction with supply-reserve distribution K.*

We prove this auxiliary complete-information result by studying an auxiliary problem in which a bidder's bid satisfies a best-response first order condition but is not necessarily a best response to the random elastic supply and other bidders' mixed strategies. We show that if—counterfactually—the seller was able to set the random supply-reserve distribution separately for this focal bidder, holding the other bidders' behavior fixed, then the seller would optimize this part of the revenue by keeping the quantity allocated to the focal bidder constant and randomizing only over reserve prices. That is, analyzing constant supply and random reserve decouples the focal bidder's best response from strategies of other bidders. Thus, given the symmetry of the problem, the seller is able to implement such a revenue maximizing scheme via a pay-as-bid auction with fixed supply and the same random reserve distribution for all bidders. Leveraging the simplification brought by being able to restrict attention on random reserve only, we bound the maximum revenue of the seller by the revenue from a deterministic supply and deterministic reserve pay-as-bid (and uniform-price with identical supply and reserve).

Having shown that if the seller knew the bidders' common information, then she can do no better than set deterministic elastic supply so as to maximize the revenue, it remains to observe that the seller can obtain this revenue pointwise with an elastic supply curve. This observation relies on the following notion of regularity.

Definition 1. [Regular Demand] Let $\mathcal{S} = \{(p^\star, q^\star) \colon \exists s,\ p^\star \in \arg\max_p p v^{-1}(p; s),\ q^\star = v^{-1}(p; s)\}$ be the set of optimal monopoly prices. Bidder values are *regular* if, for any $(p, q), (p', q') \in \mathcal{S}$, the inequality $p' < p$ implies $q' < q$.

[52] Theorem 8 extends to mixed strategies.

Values are regular if the monopolist's optimal price and quantity are in monotone correspondence. When values are increasing in signal s (an assumption we do not impose), demand is regular when $p + v^{-1}(p;s)/v_p^{-1}(p;s)$ is increasing in s. Thus our regularity condition is similar to the regularity condition in [Myerson, 1981]. When bidder values are regular the seller can implement optimal reserve and quantity via an elastic supply function even though the seller does not know the bidders' information.

Theorem 9. **[Deterministic Auctions Are Optimal]** *When bidder values are regular then revenue in the pay-as-bid auction is maximized by implementing a deterministic supply curve. Any mixed-strategy equilibrium of the pay-as-bid auction with any random elastic supply raises weakly lower revenue than the unique equilibrium of the pay-as-bid auction with optimal deterministic supply.*

Because deterministic elastic supply is not only optimal in pay-as-bid, but also extracts the same revenue as if the seller knew bidders' values (but could only set a price), we can also conclude the following:

Theorem 10. **[Pay-as-Bid Revenue Dominance]** *If bidder values are regular then the unique equilibrium of the optimal pay-as-bid auction raises weakly more revenue than any mixed-strategy equilibrium in uniform-price auction with any supply-reserve distribution.*

Furthermore, for a generic distribution of values there are multiple equilibria in uniform-price, and the revenue in a generic uniform-price equilibrium is strictly lower than the revenue in optimal pay-as-bid. This last point follows from the underpricing equilibrium constructions in, e.g.. Back and Zender [1993] and LiCalzi and Pavan [2005].

Finally, our analysis of optimal elastic supply implies that an analogue of the information disclosure Theorem 6 remains true under random elastic supply. Recall that in this theorem the quantity is exogenously realized and the seller has the ability to communicate this cap to the bidders. Because the optimal elastic supply is constructed point-by-point and does not depend on the quantity cap other than in the inelastic part of the supply where the cap is binding, in the current elastic supply setting the seller still wants to set the elastic supply (where possible) and fully reveal their private information.

Theorem 11. **[Optimality of Information Disclosure with Elastic Supply]** *If the bidders' values are regular then the seller's expected revenue is maximized when the seller commits to fully reveal the realization of the elastic supply curve.*

B Transparency and Noncompetitive Demand

As an application of our analysis, note that multi-unit auctioneers frequently obtain revenue not only from competitive bidders but also from noncompetitive bidders who pay a fixed price determined by the auction's outcome. For example, in France [Agence France Tr\{}'esor, 2022], the Czech Republic [Ministry of Finance, 2016], , and Korea [Ministry of Economics, 2021] noncompetitive bidders receive supply in addition to the supply that is auctioned to competitive bidders. When the price paid by noncompetitive bidders is monotone in the auction's market-clearing price, our Theorem 5 remains valid.

Corollary 7. [Transparency of Optimal Supply with Noncompetitive Demand] *If the seller sets the distribution of the supply in the auction and the noncompetitive bidders pay a per-unit price that is weakly increasing in the auction's market-clearing price, then the sum of the seller's revenue from competitive and noncompetitive bidders is maximized by setting deterministic supply in the auction.*

In Corollary 7 we allow the noncompetitive demand Q_{nc} to be random. The corollary follows from inequality (3). Denote by $p^F(Q_c; s)$ the equilibrium market-clearing price when the bidders believe that competitive supply Q_c has distribution F, the realized supply is Q_c, and bidders' signal is s; and let $p_{nc}(p^\star)$ be the price paid by noncompetitive bidders as a function of the market-clearing price $p^\star$. Considering payments from both competitive and noncompetitive bidders, the seller maximizes $\mathbb{E}\left[\pi^F\left(Q_c; s\right) + p_{nc} \circ p^F\left(Q_c; s\right) Q_{nc}\right]$ over F. Inequality (3) provides an upper bound on competitive revenue, $\mathbb{E}\left[\pi^F(Q_c; s)\right] \leq \int_0^{\overline{Q}^R} \mathbb{E}_s\left[\pi^{\delta_x}(x; s)\right] dF(x)$, and since bids are below values Theorem 1 implies that, given a realized competitive quantity Q_c, the equilibrium market-clearing price $p^F\left(Q_c; s\right)$ is lower than the market price $p^{\delta_{Q_c}}(Q_c; s)$ when bidders with signal s know that competitive supply is Q_c. Because p_{nc} is monotone in the market-clearing price, $\int_0^{Q^R} \mathbb{E}_{s,Q_{nc}}\left[\pi^{\delta_x}\left(x; s\right) + p_{nc} \circ p^{\delta_x}\left(x; s\right) Q_{nc}\right] dF\left(x\right)$ is an upper bound on the seller's revenue, and in turn is bounded from above by $\max_{Q\in[0,Q^R]} \mathbb{E}_{s,Q_{nc}} \Big[\pi^{\delta_Q}\left(Q; s\right)$ The seller can achieve this latter upper bound by setting deterministic supply equal to $\arg\max_{Q\in[0,Q^R]} \mathbb{E}_{s,Q_{nc}}\left[\pi^{\delta_Q}\left(Q; s\right) + p_{nc} \circ p^{\delta_Q}\left(Q; s\right) Q_{nc}\right]$. Note that this same argument works when p_{nc} is stochastic and has expectation increasing in the market-clearing price. Corollary 7 and the argument remain valid irrespective of whether noncompetitive demand Q_{nc} is observed by the seller prior to setting F.

As with Theorem 5, Theorem 6—which shows that if the seller cannot affect the distribution of supply, they would still prefer to announce realized supply—extends to the case where the seller maximizes the total revenue obtained from not only from the allocation to competitive bidders who submit demand curves, but also from the noncompetitive bidders with inelastic demand. For such a seller, it remains optimal to fully reveal the realization of supply before competitive bids are submitted as long as the price paid by non-competitive bidders is a weakly increasing function of the market-clearing price.

Corollary 8. [Optimality of Information Disclosure with Noncompetitive Demand] *Suppose that noncompetitive demand is $Q_{nc} \sim F_{nc}$, and that competitive supply is $\overline{Q} - Q_{nc}$. If the seller allocates quantity Q_{nc} to noncompetitive bidders at price $p_{nc}(p^\star)$, which is weakly increasing in the market-clearing price $p^\star$, the seller's revenue is maximized when the seller commits to fully-reveal the realization of noncompetitive demand.*

The assumption that the per-unit price paid by noncompetitive bidders is increasing in the market-clearing price allows for noncompetitive demand to be filled at a fixed price, or at the market-clearing price, or at a constant markup over the market-clearing price (among other possibilities).[53] In light of Theorem 6, Corollary 8 is straightforward to prove. The seller's revenue from competitive bidders is highest when supply is announced before bids are submitted. Moreover, announcing available supply weakly increases the market-clearing price, since bids are below marginal values except at the maximum feasible quantity (1). Then announcing the realization of supply increases the expected revenue from competitive bidders, and also increases the *ex post* revenue from noncompetitive bidders.

On the other hand, the incentives of noncompetitive bidders, whose bids generate noncompetitive demand, are opposed to those of the auctioneer. The noncompetitive bidders would (if possible) commit to not reveal their bids prior to the submission of the competitive bidders' bids because the revelation of noncompetitive demand weakly increases the market-clearing price ex post, in turn increasing the per-unit price paid they pay.

C Welfare Ambiguity

The cross-auction comparison of outcomes other than revenue—e.g., bidders' payoffs and expected surplus—depends on the perfect Bayesian equilibrium played in the uniform-price auction.

Theorem 12. [Ambiguous Bidder Welfare Comparison] *If the value space is rich then the uniform-price design game admits perfect Bayesian equilibria in which the payoff of all bidder types is strictly higher and perfect Bayesian equilibria in which the payoff of all bidder types is strictly lower than in the unique equilibrium of the pay-as-bid design game.*

[53] In spot electricity markets in which the non-competitive electricity consumers pay exogenous prices which depend neither on the bids submitted nor the market clearing price in electricity auctions for suppliers, our Theorem 6 directly implies that the auctioneer wants to reveal the consumers' demand to the suppliers bidding in the auction.

The reason for this ambiguity is that the quantity sold and reserve price in optimal uniform price can be strictly higher, the same, or strictly lower than in pay as bid, depending on the equilibrium in uniform price, as we have seen in Lemma 1. If the reserve price $R^{\star\mathrm{UP}}$ in the uniform-price design game is strictly lower than the optimal pay-as-bid reserve $R^{\star\mathrm{PAB}}$ and the supply $Q^{\star\mathrm{UP}}$ in uniform price is deterministic and strictly higher than the optimal pay-as-bid supply $Q^{\star\mathrm{PAB}}$, then there is an equilibrium of uniform price in which all bidder types pay $R^{\star\mathrm{UP}}$ for each unit they buy their payoffs are strictly higher than in pay as bid. If, conversely, $R^{\star\mathrm{UP}} > R^{\star\mathrm{PAB}}$ and $Q^{\star\mathrm{UP}} < Q^{\star\mathrm{PAB}}$ then, irrespective of the equilibrium bids in uniform price, all bidder types have lower payoffs in uniform-price than in the pay-as-bid design game.[54] In the latter case, for distributions of bidders' value functions for which the solution to the monopoly problem (4) is unique (a generic property), the seller's revenue is also strictly lower in uniform price. Generically, there are thus equilibria of the uniform-price design that are strictly worse for all market participants than the essentially unique equilibrium of the pay-as-bid design game, but not vice versa (cf. Theorem 7).

[54]Ausubel et al. [2014] established that efficiency and revenue in uniform price can be higher or lower than in pay as bid depending on utility specification, assuming that the reserve price is zero and supply is unoptimized. As we study optimized auctions, there is no contradiction between the ambiguity they report and our revenue dominance, nor are our welfare comparisons are implicit in theirs. The welfare ambiguity we uncover is driven by equilibrium selection and obtains for all utility specification in every model with rich values. In contrast, they provide examples of ambiguity that hinge on comparing equilibria between different model specifications and that rely on ex-ante asymmetries between bidders.

Supplementary Appendix (For Online Publication): Proofs

D Proof of Theorem 1 and Auxiliary Lemmas

D.1 Proof of Theorem 1 (Minimum Market Price)

We allow mixed strategies and parameterize bidder i's mixed strategy by mixing type ξ_i; denote by $\xi = (\xi_j)_{j=1}^n$ the profile of all bidders' mixing types. As discussed at the beginning of Section 4, we hold the common signal s fixed and therefore suppress it from notation. Thus a bid is a function $b^i : [0, \overline{Q}] \times \operatorname{Supp} \xi_i \to \mathbb{R}_+$. Denote $G^i(q; b^i) = \Pr(q^i \leq q | b^i)$; that is, $G^i(q; b^i)$ is the probability that the quantity agent i receives is weakly lower than q (when submitting bid b^i in the equilibrium considered).

The (essential) minimum market clearing price $\underline{p}$ and (essential) maximum receivable quantity $\overline{q}^i$, conditional on strategy profile $(b^j)_{j=1}^n$, are defined as follows[55]

$$\underline{p} = \operatorname*{ess\,inf}_{Q,\xi} p\left(Q; \left(b^j\left(\cdot; \xi_j\right)\right)_{j=1}^n\right);$$
$$\overline{q}^i\left(\xi_i\right) = \operatorname*{ess\,sup}_{Q,\xi_-} q^i\left(Q; b^i\left(\cdot; \xi_i\right), b^{-i}\left(\cdot, \xi_{-i}\right)\right);$$
$$\underline{b}^i\left(\xi_i\right) = \lim_{q \nearrow \overline{q}^i(\xi_i)} b^i\left(q; \xi_i\right).$$

We proceed in steps.

Lemma 3. *Let $(b^j)_{j=1}^n$ be an equilibrium bid profile. If $b^i(\cdot; \xi_i)$ is a best response to $(b^j)_{j \neq i}$ and $\underline{b}^i(\xi_i) < v(\overline{q}^i(\xi_i))$, then $\overline{q}^i(\xi_i) > \inf\{q \colon b^i(q; \xi_i) = \underline{b}^i(\xi_i)\}$;that is, bidder i's bid is flat in some left neighborhood of $\overline{q}^i(\xi_i)$.*

Proof. We consider two cases in turn. First, we show that if there is an opponent j whose bid b^j has bounded slope with ξ_i-positive probability at $\overline{q}^j$ (that is for $q < \overline{q}^j(\xi_j)$ and close to $\overline{q}^j(\xi_j)$, $b^j(q; \xi_j) - b^j(\overline{q}^j(\xi_j); \xi_j) \leq M_b \left|q - \overline{q}^j(\xi_j)\right|$ for some $M_b \in \mathbb{R}$ and mass $\pi_j > 0$ of ξ_j), then $\underline{b}^i(\xi_i) = v^i(\overline{q}^i(\xi_i))$. For $\lambda > 0$ consider a deviation b^λ,

$$b^\lambda\left(q\right) = \begin{cases} b^i\left(q; \xi_i\right) & \text{if } b^i\left(q; \xi_i\right) \geq \underline{b}^i\left(\xi_i\right) + \lambda, \\ \underline{b}^i\left(\xi_i\right) + \lambda & \text{otherwise.} \end{cases}$$

[55] The essential infimum is the highest value a random variable exceeds with probability one, $\operatorname{ess\,inf}_X f(X) = \sup\{y \colon \Pr(f(X) \geq y) = 1\}$. Similarly, $\operatorname{ess\,sup}_X f(X) = \inf\{y \colon \Pr(f(X) \leq y) = 1\}$.

Let $\check{q}^\lambda = \inf\{q\colon \underline{b}^i(\xi_i) + \lambda > b^i(q;\xi_i)\}$ be the lowest quantity at which the deviation b^λ diverges from $b^i(\cdot;\xi_i)$. For $q \in [\check{q}^\lambda, \overline{q}^i(\xi_i)]$ let $\delta(q) = [\underline{b}^i + \lambda] - b^i(q;\xi_i)$ be the amount by which the deviation increases the bid. Since the slope of opponent j's bid is bounded above by M_b with probability π_j, the extra quantity allocated to bidder i when they deviate to b^λ is at least $\delta(q)/M_b$ with probability π_j where q is the allocation i would have received bidding $b^i(\cdot;\xi_i)$. For the deviation to not be profitable, it must be that the increase in payment is higher than the utility gain from additional quantities that is

$$\int_{\check{q}^\lambda}^{\overline{q}^i(\xi_i)} \int_{\check{q}^\lambda}^{q} \delta(x)\, dx dG^i\left(q; b^i(\cdot;\xi_i)\right) \geq \pi_j \mu \int_{\check{q}^\lambda}^{\overline{q}} \frac{\delta(q)}{M_b} dG^i\left(q; b^i(\cdot;\xi_i)\right),$$

where μ is a constant bound on the marginal utility of additional quantity; we may assume $\mu > 0$ since marginal values are Lipschitz continuous. Because both sides are zero at $\lambda = 0$ and are differentiable in λ, the above inequality implies the following inequality between the derivatives with respect to λ of both sides:

$$\int_{\check{q}^\lambda}^{\overline{q}^i(\xi_i)} \left(q - \check{q}^\lambda\right) dG^i\left(q; b^i(\cdot;\xi_i)\right) \geq \frac{\pi_j \mu}{M_b}\left(1 - G^i\left(\check{q}^\lambda; b^i(\cdot;\xi_i)\right)\right).$$

The left-hand side is bounded by

$$\int_{\check{q}^\lambda}^{\overline{q}^i(\xi_i)} \left(q - \check{q}^\lambda\right) dG^i\left(q; b^i(\cdot;\xi_i)\right) = \int_{\check{q}^\lambda}^{\overline{q}^i(\xi_i)} \left(1 - G^i\left(q; b^i(\cdot;\xi_i)\right)\right) dq \leq \left(\overline{q}^i(\xi_i) - \check{q}^\lambda\right)\left(1 - G^i\left(\check{q}^\lambda; b^i(\cdot;\xi_i)\right)\right).$$

Then the necessary inequality for b^λ to not be profitable implies that for $\lambda > 0$ sufficiently small,

$$\overline{q}^i(\xi_i) - \check{q}^\lambda \geq \frac{\mu\pi_j}{M_b}.$$

In particular, $\overline{q}^i(\xi_i) > \lim_{\lambda \searrow 0} \check{q}^\lambda$ and the lemma is proven in the first case.

The remaining case is that, for all opponents $j \neq i$ and all bounds M_b, the event that the slope of $b^j(\cdot;\xi_j)$ at $\overline{q}^j(\xi_j)$ is bounded above by M_b has ξ_j-probability zero. Since the bids of any bidder j are infinitely steep at $\overline{q}^j(\xi_j)$ while marginal values are Lipschitz continuous, it follows that for all opponents $j \neq i$, $\underline{b}^j(\xi_j) < v(\overline{q}^j(\xi_j))$ with ξ_j-probability one. By the previously established case of the lemma, the slope of $b^i(\cdot;\tilde{\xi}_i)$ at $\overline{q}^i(\tilde{\xi}_i)$ also cannot be bounded above by any M_b with $\tilde{\xi}_i$-positive probability. For bidder j with type ξ_j, given a quantity $\check{q} < \overline{q}^j$ define a deviation $\check{b}$ by

$$\check{b}(q) = \begin{cases} b^j(q;\xi_j) & \text{if } q < \check{q}, \\ b^j(\check{q};\xi_j) & \text{otherwise.} \end{cases}$$

Letting $\delta(q) = b^j(\check{q};\xi_j) - b^j(q;\xi_j)$, the extra expected cost associated with this deviation is bounded above by

$$\int_{\check{q}}^{\overline{q}^j(\xi_j)} \int_{\check{q}}^{q} \delta(x)\, dx dG^j\left(q; b^j(\cdot;\xi_j)\right) = \int_{\check{q}}^{\overline{q}^j(\xi_j)} \delta(q) \left(1 - G^i\left(q; b^j(\cdot;\xi_j)\right)\right) dq.$$

The extra expected utility associated with this deviation is bounded below by

$$\mu \int_{\check{q}}^{\overline{q}^j(\xi_j)} (q - \check{q})\, dG^j\left(q; b^j(\cdot;\xi_j)\right) = \mu \int_{\check{q}}^{\overline{q}^j(\xi_j)} \left(1 - G^j\left(q; b^j(\cdot;\xi_j)\right)\right) dq,$$

where μ is a constant bound on the marginal utility of additional quantity (as above). Since by definition $\lim_{\check{q}\nearrow\overline{q}^j(\xi_j)} b^j(\check{q};\xi_j) = \underline{b}^j(\xi_j)$, we infer that $\delta(q)$ is arbitrarily small for $\check{q}$ sufficiently close to $\overline{q}^j(\xi_j)$. Because $\mu > 0$ is constant, for $\check{q}$ near $\overline{q}^j(\xi_j)$ the upper bound of the expected cost of the deviation is strictly below the lower bound of the expected benefit of the deviation, hence the deviation is profitable. This contradicts that $(b^j)_{j=1}^n$ was an equilibrium bid profile and shows that the second case of the proof cannot arise, thereby concluding the proof. □

Lemma 4. *For every bidder i, in equilibrium* $\Pr(\overline{q}^i(\xi_i) > \inf\{q\colon b^i(q;\xi_i) = \underline{b}^i(\xi_i)) = 0$ *(that is flats in the left neighborhood of $\overline{q}^i(\xi_i)$ have probability 0).*

Proof. Note first that there is at most one bidder for whom $\Pr(\overline{q}^i(\xi_i) > \inf\{q\colon b^i(q;\xi_i) = \underline{b}^i(\xi_i)) > 0$, otherwise standard tie-breaking logic implies that each of the (multiple) such bidders has an incentive to slightly increase their bid at the terminal flat. Then by way of establishing a contradiction, assume that bidder i is the unique bidder for whom $\Pr(\overline{q}^i(\xi_i) > \inf\{q\colon b^i(q;\xi_i) = \underline{b}^i(\xi_i)) > 0$. Then for all of bidder i's opponents $j \neq i$, Lemma 3 implies that $\Pr(\underline{b}^j(\xi_j) = v(\overline{q}^j(\xi_j))) = 1$; without loss of generality we assume that $\underline{b}^j(\xi_j) = v(\overline{q}^j(\xi_j))$ for all opponents $j \neq i$ and all types ξ_j. Because bidder i submits a flat bid with positive probability while do opponents do not, each opponent $j \neq i$ receives their maximum allocation $\overline{q}^j(\xi_j)$ with strictly positive probability. Thus, for each ξ_j, we have that $\lim_{q\nearrow\overline{q}^j(\xi_j)} (1 - G^j(q; b^j(\cdot;\xi_j))) > 0$ and there is a common lower bound for this limit, which we denote $\pi > 0$.

For bidder j with type ξ_j, and for $\check{q} < \overline{q}^j(\xi_j)$ and ε >0, define a deviation $\check{b}$ by

$$\check{b}(q) = \begin{cases} b^j(q;\xi_j) & \text{if } q < \check{q}, \\ \underline{b}^j(\xi_j) + \varepsilon & \text{if } q \geq \check{q}. \end{cases}$$

For $\varepsilon > 0$, bidder j strictly outbids bidder i's flat bid. Since $\varepsilon > 0$ may be arbitrarily small, we omit it from the expressions of cost savings. Ignoring the ε payments, this deviation saves bidder j payment whenever the allocation (under $b^j(\cdot;\xi_j)$) would have been above $\check{q}$, but it also sacrifices gross utility whenever the allocation (under $b^j(\cdot;\xi_j)$) would have been strictly between $\check{q}$ and $\overline{q}^j(\xi_j)$. The cost savings is bounded below by $\pi\int_{\check{q}}^{\overline{q}^j(\xi_j)}\delta(q)dq$, where $\delta(q) = b^j(q;\xi_j)-\underline{b}^j(\xi_j)$; the gross utility loss is bounded above by $\lim_{q\nearrow\overline{q}^j(\xi_j)}\int_{\check{q}}^{q}\int_{\check{q}}^{y}\mu(x)dxdG^j(y;b^j(\cdot;\xi_j))$, where $\mu(x) = v(x)-\underline{b}^j(\xi_j)$. Since marginal values are Lipschitz continuous, $\mu(x)\leq(\overline{q}^j(\xi_j)-x)M_v$, where M_v is the Lipschitz modulus of marginal values. Then a necessary condition for the deviation to not be profitable is that

$$\begin{aligned}\int_{\check{q}}^{\overline{q}^j(\xi_j)}\delta\left(q\right)dq\pi &\leq \lim_{q\nearrow\overline{q}^j(\xi_j)}\int_{\check{q}}^{q}\int_{\check{q}}^{y}\mu\left(x\right)dxdG^j\left(y;b^j\left(\cdot;\xi_j\right)\right)\\ &= \lim_{q\nearrow\overline{q}^j(\xi_j)}\int_{\check{q}}^{q}\mu\left(y\right)\left(\left[1-\pi\right]-G^j\left(y;b^j\left(\cdot;\xi_j\right)\right)\right)dy\\ &\leq \lim_{q\nearrow\overline{q}^j(\xi_j)}M_v\int_{\check{q}}^{q}\left(\overline{q}^j\left(\xi_j\right)-y\right)\left(\left[1-\pi\right]-G^j\left(y;b^j\left(\cdot;\xi_j\right)\right)\right)dy.\end{aligned}$$

Since $f(\cdot)$ is continuous and $\pi>0$, this is only possible if $\lim_{q\nearrow\overline{q}^j(\xi_j)}\delta(q)/(\overline{q}^j(\xi_j)-q)=0$: that is, if bidder j's bid has zero slope at $\overline{q}^j(\xi_j)$.

Thus each of bidder i's opponents is submitting an asymptotically flat bid $b^j(\cdot;\xi_j)$ near $\overline{q}^j(\xi_j)$, with ξ_j-probability one. It follows that a slight upward deviation by bidder i by some $\lambda>0$ will be profitable: the deviation has cost bounded by $\lambda\overline{Q}$, and gains proportional to λ/M_b, where $M_b>0$, the Lipschitz upper bound on the slope of other bidders at $\overline{q}^j(\xi_j)$, may be taken to be arbitrarily small. □

When bidder i's opponents play strategies $(b^j)_{j\neq i}$ let BR_i be the set of bidder i's best responses. Define the closure of the set of bidder i's best responses to be

$$\operatorname{Cl}BR_i=\left\{b\colon\forall\varepsilon>0,\forall q\geq0\;\exists\tilde{b}\in BR_i\text{ s.t. }G^i\left(q;b\right)<1\implies\left|b\left(q\right)-\tilde{b}\left(q\right)\right|<\varepsilon\right\}.$$

To simplify exposition, to any bidding strategy $\beta\in\operatorname{Cl}BR_i$ we assign ξ_i such that $b^i(\cdot;\xi_i)\equiv\beta$. For such $b^i(\cdot;\xi_i)$ in the closure we are neither requiring that they are best responses nor that they are part of the mixing by bidder i. Relatedly, we apply the above definitions of $\overline{q}^i(\xi_i)$ and $\underline{b}^i(\xi_i)$ to such bids $b^i(\cdot;\xi_i)$ from the closure.

Lemma 5. *If $b^i(\cdot;\xi_i)$ is in the closure of the set of best responses for bidder i, then $\underline{b}^i(\xi_i)=v(\overline{q}^i(\xi_i))$.*

Proof. Suppose otherwise. Then $\overline{q}^i(\xi_i)<v^{-1}(\underline{b}^i(\xi_i))$. Lemmas 3 and 4 together imply that with ξ_i'-probability 1, $\underline{b}^i(\xi_i')=v(\overline{q}^i(\xi_i'))$ and $\inf\{q\colon b^i(q;\xi_i)=\underline{b}^i(\xi_i)\}<\overline{q}^i(\xi_i)$. Thus bidder i's maximum quantity $\overline{q}^i$ drops discontinuously at the limit $b^i(\cdot;\xi_i)$ and the only way this can happen is if there is some opponent whose bid may be arbitrarily flat. Hence there is some bidder $j \neq i$ for whom $b^j(\cdot;\xi_j)$ is in the closure of the set of best responses and $\overline{q}^j(\xi_j) < v^{-1}(\underline{b}^j(\xi_j))$.

Note that either $b^i(\cdot;\xi_i)$ is not a best response, in which case it is played with probability zero, or it is a best response and by Lemma 4 it is played with probability zero (since $\underline{b}^i(\xi_i) < v(\overline{q}^i(\xi_i))$; in either case, $b^i(\cdot;\xi_i)$ is played with probability zero. Then since $b^i(\cdot;\xi_i)$ is in the closure of the support of bidder i's best responses, for all $\varepsilon > 0$ and all $q \in (\overline{q}^i(\xi_i), v^{-1}(\underline{b}^i(\xi_i)))$ (which is non-empty) there is some type $\xi_i' \neq \xi_i$ such that $b^i(\cdot;\xi_i') \in BR_i$ is a best response to $(b^j)_{j\neq i}$ and $b^i(q;\xi_i') < \underline{b}^i(\xi_i) + \varepsilon$ and $G^i(q;b^i(\cdot;\xi_i')) < 1$; that is, quantity q is obtainable with positive probability under bid $b^i(\cdot;\xi_i')$, and the bid for this quantity is not too far above $\underline{b}^i(\xi_i)$.

For $q > \overline{q}^i(\xi_i)$, let $\gamma = q - \overline{q}^i(\xi_i)$. For type ξ_i' to obtain quantity q under bid $b^i(\cdot;\xi_i')$ with $b^i(q;\xi_i') < \underline{b}^i(\xi_i) + \varepsilon$, it must be that some opponent j's inverse bid at price $\underline{b}^i(\xi_i) + \varepsilon$ is $\varphi^j(\underline{b}^i(\xi_i) + \varepsilon;\xi_j) \leq v^{-1}(\underline{b}^i(\xi_i)) - \gamma/(n-1)$. Since γ may take any value between 0 and $\overline{q}^i(\xi_i') - \overline{q}^i(\xi_i)$, and $\varepsilon > 0$ may be arbitrarily small, it follows that when bidder i wins quantity q the quantity is won against at least one opponent with an arbitrarily flat bid. That is, as $\varepsilon > 0$ becomes small the residual supply faced by bidder i becomes infinitely elastic.

Finally, since $\inf\{\tilde{q}\colon b^i(\tilde{q};\xi_i') \leq \underline{b}^i(\xi_i')\} = \overline{q}^i(\xi_i)$ and $b^i(\tilde{q};\xi_i') \leq v(\tilde{q})$ for all $\tilde{q} \in (0, \overline{q}^i(\xi_i))$, for any $\varepsilon > 0$ there is some $q > \overline{q}^i(\xi_i')$ and type ξ_i' such that $b^i(\tilde{q};\xi_i') > b^i(q;\xi_i')$ for all $\tilde{q} < q$. Given such a q and ξ_i', fix $\lambda > 0$, define $\check{q} = \sup\{\tilde{q}\colon b^i(\tilde{q};\xi_i') \geq b^i(q;\xi_i') + \lambda\}$ and consider a deviation b^λ given by

$$b^\lambda(\tilde{q}) = \begin{cases} b^i(\tilde{q};\xi_i') & \text{if } \tilde{q} \notin [\check{q}(\lambda), q], \\ b^i(q;\xi_i') + \lambda & \text{if } \tilde{q} \in [\check{q}(\lambda), q]. \end{cases}$$

This deviation has costs equal to

$$\int_{\check{q}(\lambda)}^{\overline{q}^i(\xi_i')} \int_{\check{q}(\lambda)}^{\min\{\tilde{q},q\}} \delta(y)\, dy dG^i\left(\tilde{q}; b^i(\cdot;\xi_i')\right) = \int_{\check{q}(\lambda)}^{\overline{q}^i(\xi_i')} \delta(\min\{\tilde{q},q\}) \left(1 - G^i\left(\tilde{q}; b^i(\cdot;\xi_i')\right)\right).$$

Its benefits are bounded below by

$$\int_{\check{q}(\lambda)}^{q} \int_{\tilde{q}}^{\min\{\tilde{q}+\delta(\tilde{q})/M_b,q\}} v(y)\, dy dG^i\left(\tilde{q}; b^i(\cdot;\xi_i')\right).$$

Because the "inducing the flat" opponent's bid is arbitrarily flat (for ε small), the benefits may be bounded below again by

$$\int_{\check{q}(\lambda)}^{q} (q-\tilde{q})\, \mu dG^i\left(\tilde{q}; b^i(\cdot;\xi_i')\right) = (q - \check{q}(\lambda))\, \mu \left(1 - G^i\left(\check{q}(\lambda); b^i(\cdot;\xi_i')\right)\right) - \mu \int_{\check{q}(\lambda)}^{q} \left(1 - G^i\left(\tilde{q}; b^i(\cdot;\xi_i')\right)\right) d\tilde{q}.$$

For λ sufficiently small this deviation is profitable, hence we obtain a contradiction. $\square$

Lemma 6. *Let $(b^i)_{i=1}^n$ be a mixed-strategy equilibrium in which each for each bidder i and each bid $b^i(\cdot;\xi_i)$ in the support of bidder i's mixed strategy, $b^i(\cdot;\xi_i)$ is a best response to $(b^j)_{j\neq i}$. Then, for any signal s and profile ξ of mixing types, the market clearing price $p\left(\overline{Q}^R;\xi\right)$ at the effective maximum quantity $\overline{Q}^R$ is equal to the marginal value for per-capita maximum supply; that is, $p\left(\overline{Q}^R;\xi\right) = v\left(\frac{1}{n}\overline{Q}^R\right)$.*

Proof. Bids will be below values for all relevant quantities, thus we know that $p(\overline{Q}^R;\xi) \leq v(\overline{Q}^R/n;\xi)$ when mixed strategies are supported by best responses. Now, suppose that there is a type profile ξ such that $p(\overline{Q}^R;\xi) < v(\overline{Q}^R/n;\xi)$. By Lemmas 3 and 4, $\underline{b}^i(\xi_i) = v(\overline{q}^i(\xi_i);\xi_i)$ for all bidders i with ξ_i-probability 1, hence $p(\overline{Q}^R;\xi) < v(\overline{Q}^R/n;\xi)$ only if there is some bidder i and bid $b^i(\cdot;\xi_i') \in \operatorname{Cl} BR_i$ such that $p(\overline{Q}^R;\xi) < v(\overline{q}^i(\xi_i))$. This contradicts Lemma 5, hence it must be that $p(\overline{Q}^R;\xi) = v(\overline{Q}^R/n)$ whenever $(b^i(\cdot;\xi_i))_{i=1}^n$ is a bid profile where each $b^i(\cdot;\xi_i)$ is a best response to $(b^j)_{j\neq i}$. $\square$

Theorem 1 follows from Lemma 6 because in a mixed-strategy equilibrium, the set of bid functions $b^i(\cdot;\xi_i)$ in the support of b^i which are not best responses to $(b^j)_{j\neq i}$ has probability zero.

D.2 Pure strategy equilibrium derivation with symmetric bidder information

In this section we present the lemmas for our results on existence, uniqueness, and bid representation of pure strategy equilibria under symmetric bidder information. The argument for deterministic supply was given in the main text, and here we focus on random supply. As in the main text, to simplify notation we write $v(q)$ in lieu of $v(q;s)$ and $b^i(q)$ in lieu of $b^i(q;s)$.

Let us fix a pure-strategy candidate equilibrium $(b^i)_{i=1}^n$. Recall that bid functions are weakly decreasing and (where useful) we may assume that they are right continuous. Given equilibrium bids the market price (that is, the stop-out price) $p(Q)$ is a function of realized supply Q. In line with Appendix D.1, denote $G^i(q;b^i) = \Pr(q^i \leq q|b^i)$, and denote the inverse hazard rate of aggregate supply by $H = \frac{1-F}{f}$.

Our statements in the following results are generally about relevant quantities, such that $G^i(q;b^i) < 1$. For each bidder we ignore quantities larger than the maximum quantity this bidder can obtain in equilibrium; for instance, in the following lemmas, all bidders could submit identical flat bids above their values for units they never obtain. Correspondingly, we say that a *price* level p is *relevant* if p is strictly higher than $\underline{p}$ and weakly below the highest bid.

Lemma 7. *For no relevant price level p are there two or more bidders who, in equilibrium, bid constant value p flat on some non-trivial intervals of quantities.*

Proof. The proof resembles similar proofs in other auction contexts. Suppose agent i bids p on $(q_{i\ell}, q_{ir})$ and bidder j bids p on $(q_{j\ell}, q_{jr})$ and these quantities are relevant. Since the support of supply is $\left[0, \overline{Q}\right]$, it must be that $G^i(q_{ir}; b^i) > G^i(q_{i\ell}; b^i)$ and $G^j(q_{jr}; b^j) > G^j(q_{\ell j}; b^j)$. Let $\bar{q}^i = \mathbb{E}_Q[q^i|p(Q) = b(q_{ir})]$; without loss of generality, we may assume that agent i is such that $\bar{q}^i < q_{ir}$. If $v^i(\bar{q}^i) < b^i(q_{ir})$, the agent has a profitable downward deviation. The agent also has a profitable deviation if $v^i(\bar{q}^i) \geq b^i(q_{ir})$: she can increase her bid slightly by $\lambda > 0$ on $[q_{i\ell}, q_{ir})$ (enforcing monotonicity constraints as necessary to the left of $q_{i\ell}$), keeping her bid below value if necessary. □

Lemma 8. *Bids are below values: $b^i(q) \leq v^i(q)$ for all relevant quantities, and $b^i(q) < v^i(q)$ for $q < \varphi^i(p(\overline{Q}))$.*

Proof. Suppose that there exists q with $b^i(q) > v^i(q)$; because b^i is monotonic and v^i is continuous, there must exist a range $(q_{i\ell}, q_{ir})$ of relevant quantities such that $b^i(q) > v^i(q)$ for all $q \in (q_{i\ell}, q_{ir})$. The agent wins quantities from this range with positive probability, and hence the agent could profitably deviate to

$$\hat{b}^i(q) = \min\left\{b^i(q), v^i(q)\right\}.$$

Such a deviation never affects how she might be rationed, by the first part of this proof; hence it is necessarily utility-improving.

Now consider $q < \varphi^i\left(p\left(\overline{Q}\right)\right)$. If $b^i(q) = v^i(q)$ then monotonicity of b^i and Lipschitz-continuity of v^i imply that for small $\varepsilon > 0$ winning units $[q - \varepsilon, q]$ brings per unit profit lower than $M\varepsilon$, where M is the Lipschitz modulus of v. By lowering the bid for quantities $q' \in [q - \varepsilon, q + \varepsilon]$ to $\hat{b}^i(q') = \min\{v^i(q) - \varepsilon, b^i(q')\}$, the utility loss from losing the relevant quantities is at most $2M\varepsilon^2\left(G_i\left(q + \varepsilon; b^i\right) - G_i\left(q - \varepsilon; b^i\right)\right)$. Notice that the right-hand probability difference goes to zero as ε goes to zero. At the same time the cost savings from paying lower bids at quantities higher than $q + \varepsilon$ is (at least) of order ε^2. Hence this deviation is profitable, and it cannot be that $b^i(q) = v^i(q)$.

□

Lemma 9. *The market clearing price $p(Q)$ is strictly decreasing in supply Q on $\left[0, \overline{Q}^R\right]$.*

Proof. We show first that the market clearing price is strictly decreasing in supply for all Q such that $p(Q) > \inf_{Q'} p(Q') = \underline{p}$. We then show that p is strictly decreasing at $\overline{Q}^R$ as long as for any bidder i residual supply $\sum_{j\neq i} \varphi^j(\cdot)$ has nonzero slope at $\underline{p}$. Since Theorem 1 shows that it is without loss of generality to assume that $b^i(\overline{q}^i) = v(\overline{Q}^R/n; s)$, Lemma 8 shows that bids are below values, and values are Lipschitz continuous, it follows that residual supply has nonzero slope at $\underline{p}$, and therefore the market clearing price is strictly decreasing in Q.

Since bids are weakly decreasing in quantity, the market price is weakly decreasing as a direct consequence of the market-clearing equation. If price is not weakly decreasing in quantity at some Q, then a small increase in Q will not only increase the price, but will weakly decrease the quantity allocated to each agent. This implies that total demand is no greater than Q, contradicting market clearing.

Lemma 7 is sufficient to imply that the market price must be strictly decreasing for all Q such that $p(Q) > \underline{p}$: at every price level at which at least two bidders pay with positive probability for some quantity, at most one of the submitted bid functions is flat (that is there is an interval of quantities at which the bid equals this price). Furthermore, for no price level $p > \underline{p}$ that with positive probability a bidder pays for some quantity, we can have exactly one bidder, i, submitting a flat bid at price p on an interval of relevant quantities. Indeed, in equilibrium bidder i cannot benefit by slightly reducing the bid on this entire interval; thus it must be that there is some other agent j whose bid function is right continuous at price p. If $p = 0$, all opponents $j \neq i$ have a profitable deviation. If $p > 0$, we appeal to Lemma 8. Given that i submits a flat bid and the bids of bidder j are strictly below her values for some non-trivial subset of quantities at which her bid is near p, bidder j can then profit by slightly raising her bid; this reasoning is similar to that given in the proof of Lemma 7.

We now show that $p(\cdot)$ is strictly decreasing for all Q. Otherwise, following Lemma 7, there is a bidder i who is submitting a flat bid at $\underline{p}$. Denote the left end of this bidder's flat by $\underline{q}_i = \inf\{q\colon b^i(q) = \underline{p}\}$; by assumption, $\underline{q}_i < \overline{q}_i$. (To see that $\overline{q}_i > 0$ one might also note that otherwise bidder i would almost surely receive 0 utility ex post, which is not possible in any equilibrium of pay as bid with symmetric bidders). Let $\varepsilon, \lambda > 0$ and define a deviation

$$\hat{b}^{\varepsilon\lambda}(q) = \begin{cases} b^i(q) & \text{if } b^i(q) > \underline{p} + \lambda, \\ \underline{p} + \lambda & \text{if } b^i(q) \leq \underline{p} + \lambda \text{ and } q \leq \underline{q}_i + \varepsilon, \\ \underline{p} & \text{otherwise.} \end{cases}$$

That is, $\hat{b}^{\varepsilon\lambda}$ is b^i, with λ added for length ε at $\underline{q}_i$, and adjusting for the fact that bids must be monotone decreasing. Note that this deviation increases costs by at most $(\varepsilon + (\underline{q}_i - \varphi^i(\underline{p} + \lambda)))\lambda$, with at most probability one. When $q_i \in [\underline{q}_i, \underline{q}_i + \varepsilon]$, it increases the quantity allocation to (approximately) $\max\{\underline{q}_i+\varepsilon, q+\lambda M\}$, where M is the slope of residual supply at the minimum price, $M = \left|\sum_{j\neq i}\varphi^j_p(\underline{p})\right|$.[56] Let $\mu \equiv v^i(\underline{q}_i+\varepsilon)-(\underline{p}+\lambda)$; since bids are below values and values are strictly decreasing, $\mu > 0$ when ε and λ are sufficiently small. Then for the deviation to be nonoptimal, it must be that

$$\left(\varepsilon+\left(\underline{q}_i-\varphi^i\left(\underline{p}+\lambda\right)\right)\right)\lambda \geq \mathbb{E}\left[\left(\max\left\{\varepsilon, q+\frac{\lambda}{M}\right\}-q\right)\mu\middle| q\in\left[\underline{q}_i,\underline{q}_i+\varepsilon\right]\right]$$
$$=\mathbb{E}\left[\left(\max\left\{\varepsilon-q, \frac{\lambda}{M}\right\}\right)\mu\middle| q\in\left[\underline{q}_i,\underline{q}_i+\varepsilon\right]\right].$$

Letting $\overline{Q}_{-i}=\sum_{j\neq i}\overline{q}_j$, this can be rewritten as

$$\left(\varepsilon+\left(\underline{q}_i-\varphi^i\left(\underline{p}+\lambda\right)\right)\right)\lambda\int_{\underline{q}_i}^{\underline{q}_i+\varepsilon}dF\left(q+\overline{Q}_{-i}\right)\geq\int_{\underline{q}_i}^{\underline{q}_i+\varepsilon}\max\left\{\varepsilon+\underline{q}_i-q,\frac{\lambda}{M}\right\}\mu dF\left(q+\overline{Q}_{-i}\right)$$
$$\geq\int_{\underline{q}_i}^{\underline{q}_i+\varepsilon-\frac{\lambda}{M}}\frac{\mu\lambda}{M}dF\left(q+\overline{Q}_{-i}\right).$$

The $\lambda>0$ multipliers cancel; integrating through gives

$$\left(\varepsilon+\left(\underline{q}_i-\varphi^i\left(\underline{p}+\lambda\right)\right)\right)\left(F\left(\underline{q}_i+\varepsilon+\overline{Q}_{-i}\right)-F\left(\underline{q}_i+\overline{Q}_{-i}\right)\right)\geq\frac{\mu}{M}\left(F\left(\underline{q}_i+\varepsilon-\frac{\lambda}{M}+\overline{Q}_{-i}\right)-F\left(\underline{q}_i+\overline{Q}_{-i}\right)\right)$$

From here the argument is standard. For any $\varepsilon>0$ there is $\lambda>0$ such that $\varepsilon-\lambda/M\geq\varepsilon/2$ and $\underline{q}_i-\varphi^i(\underline{p}+\lambda)<\varepsilon/2$. Thus it must be that

$$\frac{3}{2}\varepsilon\left(F\left(\underline{q}_i+\varepsilon+\overline{Q}_{-i}\right)-F\left(\underline{q}_i+\overline{Q}_{i-i}\right)\right)\geq\frac{\mu}{M}\left(F\left(\underline{q}_i+\frac{1}{2}\varepsilon-\overline{Q}_{-i}\right)-F\left(\underline{q}_i+\overline{Q}_{-i}\right)\right)$$
$$\iff\quad F\left(\underline{q}_i+\varepsilon+\overline{Q}_{-i}\right)-F\left(\underline{q}_i+\overline{Q}_{-i}\right)\geq\frac{\mu}{3M}\left[\frac{F\left(\underline{q}_i+\frac{1}{2}\varepsilon-\overline{Q}_{-i}\right)-F\left(\underline{q}_i+\overline{Q}_{-i}\right)}{\frac{1}{2}\varepsilon}\right].$$

This must hold for all $\varepsilon>0$. Because $\underline{q}_i+\overline{Q}_{-i}<\overline{Q}$, supply distribution F is Lebesgue absolutely continuous near $\underline{q}_i+\overline{Q}_{-i}$; taking the limit as $\varepsilon\searrow 0$ gives

$$0\geq\frac{\mu f\left(\underline{q}_i+\overline{Q}_{-i}\right)}{3M}.$$

Since $f(\cdot)>0$ at $\underline{q}_i+\overline{Q}_{-i}$, this is a contradiction since M is finite (Lemma 12). In this case, bidder i has a profitable deviation. $\square$

[56] Because we are ultimately letting ε and λ go to zero, this approximation is sufficient. Formally, we may consider $M'<M$ and allow δ to be small enough that the slope of residual supply never falls below M'.

Corollary 9. *In any pure-strategy equilibrium, bid functions are strictly decreasing.*

We define the derivative of G^i with respect to b as follows. For any q and b^i, the mapping $t \mapsto G^i(q; b^i + t)$ is weakly decreasing in t, and hence differentiable almost everywhere. With some abuse of notation, whenever it exists we denote the derivative of this mapping with respect to t by $G^i_b(q; b^i)$.

Lemma 10. *For each agent i and almost every q we have:*

$$G^i_b\left(q; b^i\right) = f\left(q + \sum_{j \neq i} \varphi^j\left(b^i(q)\right)\right) \sum_{j \neq i} \varphi^j_p\left(b^i(q)\right).$$

Proof. By definition, $G^i(q; b^i) = \Pr(q^i \leq q | b^i)$. From market clearing, this is

$$\begin{aligned} G^i\left(q; b^i\right) =& \Pr\left(Q \leq q + \sum_{j \neq i} \varphi^j\left(b^i(q)\right)\right) \\ =& F\left(q + \sum_{j \neq i} \varphi^j\left(b^i(q)\right)\right). \end{aligned}$$

Where the demands φ^j of agents $j \neq i$ are differentiable, we have

$$G^i_b\left(q; b^i\right) = f\left(q + \sum_{j \neq i} \varphi^j\left(b^i(q)\right)\right) \sum_{j \neq i} \varphi^j_p\left(b^i(q)\right).$$

Since for all j, the demand function φ^j must be differentiable almost everywhere, the result follows. $\square$

Lemma 11. *At points where $G^i_b(q; b^i)$ is well-defined, the first-order conditions of the pay-as-bid auction are given by*

$$-\left(v(q) - b^i(q)\right) G^i_b\left(q; b^i\right) = 1 - G^i\left(q; b^i\right).$$

In the case of pure strategies under symmetric bidder information,[57] *the first-order condition can be written as*

$$-\left(v(q) - b^i(q)\right)\left(\frac{d}{db} Q\left(b^i(q)\right) - \varphi^i_p\left(b^i(q)\right)\right) = H\left(Q\left(b^i(q)\right)\right),$$

[57]The definition of the derivative of bidder i's distribution of supply, G^i_b, obtained in Lemma 10, assumes pure strategies under symmetric bidder information. The first order condition derived here is invariant to the source of randomness in the bidder's allocation, but the statement in terms of aggregate demand holds only for pure strategies under symmetric bidder information.

where $Q(p)$ is the inverse of $p(Q)$.

Proof. The agent's maximization problem is given by

$$\max_b \int_0^{\overline{Q}} \int_0^q v(x) - b(x)\, dx dG^i(q; b).$$

Integrating by parts, we have

$$\max_b - \left[\left(1 - G^i(q; b) \right) \int_0^q v(x) - b(x)\, dx \right] |_{q=0}^{\overline{Q}} + \int_0^{\overline{Q}} (v(q) - b(q)) \left(1 - G^i(q; b) \right) dq.$$

In the first square bracket term, both multiplicands are bounded for $q \in [0, \overline{Q}]$, hence the fact that $1 - G^i(\overline{Q}; b) = 0$ for all b and $\int_0^0 v(x) - b(x) dx = 0$ for all b allows us to restate the agent's optimization problem as

$$\max_b \int_0^{\overline{Q}} (v(q) - b(q)) \left(1 - G^i(q; b) \right) dq,$$

where the integral still equals bidder's expected utility from bidding b. The calculus of variations gives us the necessary condition

$$- \left(1 - G^i \left(q; b^i \right) \right) - \left(v(q) - b^i(q) \right) G_b^i \left(q; b^i \right) = 0.$$

This holds at almost all points at which G_b^i is well-defined. Rearrangement yields the first expression for the first-order condition.

To derive the second expression, let us substitute into the above formula for G^i and G_b^i from the Lemma 10. We obtain

$$- \left(v(q) - b^i(q) \right) f \left(q + \sum_{j \neq i} \varphi^j \left(b^i(q) \right) \right) \left(\sum_{j \neq i} \varphi_p^j \left(b^i(q) \right) \right) = 1 - F \left(q + \sum_{j \neq i} \varphi^j \left(b^i(q) \right) \right),$$

Now, $Q(p)$ is well-defined since we have shown that p is strictly monotone. By Corollary 9 bids are strictly monotone in quantities and hence $q + \sum_{j \neq i} \varphi^j(b^i(q)) = Q(b^i(q))$, and

$$- \left(v(q) - b^i(q) \right) \left(\sum_{j \neq i} \varphi_p^j \left(b^i(q) \right) \right) = H \left(Q \left(b^i(q) \right) \right).$$

Since $\sum_{j \neq i} \varphi_p^j(b^i(q)) = \frac{d}{db} Q(b^i(q)) - \varphi_p^i(b^i(q))$, the second expression for the first order condition obtains. □

Lemma 12. *Each bidder's equilibrium inverse bid is Lipschitz continuous at all prices p at* *which the bidder receives a quantity in* $[0, q^i(\overline{Q}))$.

Proof. Consider an equilibrium bid profile $(b^i)_{i=1}^n$, and let $q^i(Q)$ be the resulting allocation of bidder i given supply Q. By way of contradiction, assume that bidder i's inverse bid φ^i is not Lipschitz continuous at some price p at which the bidder receives a quantity $q = \varphi^i(p)$ in $[0, q^i(\overline{Q}))$. Then $p = b^i(q)$ and $G^i(q; b^i) < 1$. Let $Q^{\min} \in \left[0, \overline{Q}\right)$ be a supply at which $q = q^i(Q^{\min})$; in particular, $Q^{\min} = q + \sum_{j \neq i} \varphi^j(b^i(q))$.

The failure of Lipschitz continuity implies that either for any $\tilde{K}$ there are arbitrarily small $\varepsilon > 0$ such that $\varphi^i(p - \varepsilon) - \varphi^i(p) > \tilde{K}\varepsilon$, or for any $\tilde{K}$ there are arbitrarily small $\varepsilon > 0$ such that $\varphi^i(p) - \varphi^i(p + \varepsilon) > \tilde{K}\varepsilon$. We provide the argument for the former case; the analysis of the latter cases is analogous.[58] In this case, for any $K > 0$, there are arbitrarily small $\varepsilon > 0$ such that

$$b^i(q) - b^i(q + \varepsilon) < K\varepsilon. \tag{5}$$

We proceed in five steps. First, we show that bidder i wins an arbitrarily large fraction of residual market quantity just above Q. Second, there exist non-trivial intervals on which bidder i wins an arbitrarily large fraction of the residual market quantity. Third, the bid of bidder i is nearly flat on non-trivial intervals just above Q. Fourth, each opponent j's bid must be steep near $q^j(Q^{\min})$. Fifth and finally, the last two claims allow us to conclude that bidder i's inverse bid must be discontinuous at p, contradicting Corollary 9 in which we showed that equilibrium bids are strictly decreasing.

Claim 1. There is a subsequence of aggregate quantities converging to $Q^{\min}$ on which i receives all additional supply beyond $Q^{\min}$; that is, for any $M < 1$ and $\bar{\varepsilon} > 0$, there is $Q \in (Q^{\min}, Q^{\min} + \bar{\varepsilon})$ such that $q^i(Q) > q + (Q - Q^{\min})M$.

Proof. Take any $\varepsilon > 0$ and consider the deviation b^ε that "kicks out" the bid function at q for length ε,

$$b^\varepsilon(q') = \begin{cases} b^i(q') & \text{if } q' \notin [q, q + \varepsilon], \\ b^i(q) = p & \text{if } q' \in [q, q + \varepsilon]. \end{cases}$$

This deviation increases payment by at most $\int_q^{q+\varepsilon} b^i(q) - b^i(x)\, dx$ whenever the realized quantity $q' > q$, which occurs with probability $1 - G^i(q; b^i) \equiv P$. It also increases the allocation: as in equilibrium the opponents bids are strictly decreasing (by Corollary 9),

[58] In the former case we maintain the assumption that b^i is right continuous. In the latter case, we consider $\hat{b}^i$, the left-continuous modification of b^i. Because bids are monotone on a compact domain, $\hat{b}^i$ and b^i agree almost everywhere and yield the same utility for bidder i, we infer that any utility-improving deviation from $\hat{b}^i$ is a utility-improving deviation from b^i, and vice-versa. As, in the latter case, φ^i fails Lipschitz continuity to the right of p, we conclude that b^i is left continuous at q, so b^i and $\hat{b}^i$ agree at this point and $\hat{\varphi}^i$ (the inverse of $\hat{b}^i$) also fails Lipschitz continuity to the right of p. We may then derive the same contradiction as in the former case.

whenever the allocation of i would have been in the interval $(q, q+\varepsilon)$, the allocation increases to $q+\min\{\varepsilon, Q-Q^{\min}\}$. The resulting gain in expected utility attributable to the allocation increase is

$$\int_{Q^{\min}}^{Q^{\max}} \int_{q^i(Q)}^{q+\min\left\{\varepsilon, Q-Q^{\min}\right\}} v(x) - b^i(q)\, dx dF(Q),$$

where $Q^{\max} = [q+\varepsilon] + \sum_{j \neq i} \varphi^j(b^i(q+\varepsilon))$. Notice that $Q^{\max} > Q^{\min} + \varepsilon$. As $(b^j)_{j=1}^n$ is an equilibrium, the costs of the deviation weakly outweigh the benefits,

$$\left[\int_q^{q+\varepsilon} b^i(q) - b^i(x)\, dx\right] P \geq \int_{Q^{\min}}^{Q^{\max}} \int_{q^i(Q)}^{q+\min\left\{\varepsilon, Q-Q^{\min}\right\}} v(x) - b^i(q)\, dx dF(Q).$$

The left-hand side is bounded from above by $[b^i(q) - b^i(q+\varepsilon)]\varepsilon P$, and the right-hand side is bounded from below by

$$\begin{aligned}
&\int_{Q^{\min}}^{Q^{\max}} \int_{q^i(Q)}^{q+\min\left\{\varepsilon, Q-Q^{\min}\right\}} v(x) - b^i(q)\, dx dF(Q) \\
&\geq \int_{Q^{\min}}^{Q^{\max}} \left(q + \min\left\{\varepsilon, Q - Q^{\min}\right\} - q^i(Q)\right) \left[v\left(q + \min\left\{\varepsilon, Q - Q^{\min}\right\}\right) - b^i(q)\right] dF(Q) \\
&\geq \left[v\left(q + \min\left\{\varepsilon, Q^{\max} - Q^{\min}\right\}\right) - b^i(q)\right] \underline{f} \int_{Q^{\min}}^{Q^{\max}} \left(q + \min\left\{\varepsilon, Q - Q^{\min}\right\} - q^i(Q)\right) dQ
\end{aligned}$$

where $\underline{f} > 0$ is a lower bound on $f(\cdot)$ on $[Q^{\min}, Q^{\max}]$; such a bound exists because f is continuous and $f(\cdot) > 0$ on $[Q^{\min}, Q^{\max}]$ for small ε (as then $Q^{\max} < \overline{Q}$).

A necessary condition for the alternate bid b^ε to not improve bidder i's utility is

$$\begin{aligned}
&\left[b^i(q) - b^i(q+\varepsilon)\right] \varepsilon P \\
&\geq \left[v\left(q + \min\left\{\varepsilon, Q^{\max} - Q^{\min}\right\}\right) - b^i(q)\right] \underline{f} \int_{Q^{\min}}^{Q^{\max}} \left(q + \min\left\{\varepsilon, Q - Q^{\min}\right\} - q^i(Q)\right) dQ \\
&= \left[v(q+\varepsilon) - b^i(q)\right] \underline{f} \int_{Q^{\min}}^{Q^{\max}} \left(q + \min\left\{\varepsilon, Q - Q^{\min}\right\} - q^i(Q)\right) dQ
\end{aligned}$$

Let $C > 0$ be such that $C \leq [v(q+\varepsilon) - b^i(q)]\underline{f}/P$; we then require

$$b^i(q) - b^i(q+\varepsilon) \geq \frac{C}{\varepsilon} \int_{Q^{\min}}^{Q^{\max}} \left(q + \min\left\{\varepsilon, Q - Q^{\min}\right\} - q^i(Q)\right) dQ. \tag{6}$$

Consider any $M \in (0,1]$ such that

$$q^i(Q) \leq q + (Q - Q^{\min})M$$

for $Q \in (Q^{\min}, Q^{\max})$; such an M trivially exists because this inequality holds for $M = 1$.

Note that $q+\varepsilon = q^i\left(Q^{\max}\right) \le q + \left(Q^{\max} - Q^{\min}\right) M$ implies that

$$Q^{\max} \ge Q^{\min} + \frac{1}{M}\varepsilon.$$

The bounds on $Q^{\max}$ and $q^i(Q)$ imply that

$$\begin{aligned}
&\int_{Q^{\min}}^{Q^{\max}} \left(q + \min\left(\varepsilon, Q - Q^{\min}\right) - q^i\left(Q\right)\right) dQ \\
&= \int_{Q^{\min}}^{Q^{\min}+\varepsilon} \left(q - q^i\left(Q\right) + Q - Q^{\min}\right) dQ + \int_{Q^{\min}+\varepsilon}^{Q^{\max}} \left(q - q^i\left(Q\right) + \varepsilon\right) dQ \\
&\ge \int_{Q^{\min}}^{Q^{\min}+\varepsilon} \left(-(Q - Q^{\min})M + Q - Q^{\min}\right) dQ \\
&= \int_{Q^{\min}}^{Q^{\min}+\varepsilon} \left(\left(1-M\right)\left(Q - Q^{\min}\right)\right) dQ = \left(1-M\right)\frac{\varepsilon^2}{2}.
\end{aligned}$$

Plugging this into the necessary condition above we transform it to

$$b^i\left(q\right) - b^i\left(q+\varepsilon\right) \ge \frac{C}{\varepsilon}\left(1-M\right)\frac{\varepsilon^2}{2} = \frac{C\left(1-M\right)}{2}\varepsilon$$

for all sufficiently small $\epsilon > 0$ and any $M \in (0,1]$ such that $q^i(Q) \le q + (Q - Q^{\min})M$ for $Q \in (Q^{\min}, Q^{\min} + \varepsilon)$.

The above bound and equation 5 jointly imply that, for any $M < 1$ and $\bar{\varepsilon} > 0$, there is $Q \in (Q^{\min}, Q^{\min} + \bar{\varepsilon})$ such that $q^i(Q) > q + (Q - Q^{\min})M$. This proves the claim: there are supply realizations arbitrarily close to $Q^{\min}$ for which agent i wins an arbitrarily large proportion of aggregate quantity above $Q^{\min}$. QED

Claim 2. For any $M < 1$ and any $\varepsilon > 0$ there is an aggregate quantity Q' and a quantity $q' = q^i(Q')$ won by bidder i such that for all $\tilde{Q}' \in (Q', Q' + \varepsilon)$,

$$q^i\left(\tilde{Q}'\right) \ge q' + \left(\tilde{Q}' - Q'\right)M.$$

Furthermore, Q' can be taken to be arbitrarily close to $Q^{\min}$.

Proof. Because $q^i(\cdot)$ is weakly increasing and $q + (Q - Q^{\min})M$ is continuous in Q, by applying Claim 1 to sufficiently larger $M < 1$, we obtain intervals $(Q', Q' + \varepsilon)$ such that for for all $\tilde{Q}' \in (Q', Q' + \varepsilon)$,

$$q^i\left(\tilde{Q}'\right) \ge q + \left(\tilde{Q}' - Q^{\min}\right)M$$

as claimed. QED

Claim 3. There is a constant $C > 0$ such that for any $M < 1$ and for any Q' from Claim 2 sufficiently close to $Q^{\min}$ and for any sufficiently small $\delta > 0$, the bids near $q' = q^i\left(Q'\right)$

satisfy

$$b^i(q') - b^i(q'+\delta) \leq C(1-M)\delta.$$

Proof. Consider M, ε, Q', and q' from Claim 2. For $\delta \in (0,\varepsilon)$ consider a deviation

$$b^\delta(\tilde{q}') = \begin{cases} b^i(q'+\delta) & \text{if } \tilde{q}' \in [q', q'+\delta], \\ b^i(\tilde{q}') & \text{otherwise.} \end{cases}$$

This deviation saves payment $\int_{q'}^{q'+\delta} b^i(x) - b^i(q'+\delta)\,dx$ with probability at least $1-G^i(q'+\delta)$, and, for δ sufficiently small, we can bound this probability from below by some constant $P>0$. In equilibrium the saved payment is weakly lower than the associated gross utility loss from winning fewer units; the latter is bounded above by $v(0)(1-M)\delta(G^i(q'+\delta)-G^i(q'))$, where $(1-M)\delta$ is the bound on quantity loss implied by the bound in Claim 2. Thus

$$P\int_{q'}^{q'+\delta} b^i(x) - b^i(q'+\delta)\,dx \leq v(0)(1-M)\left(G^i(q'+\delta)-G^i(q')\right)\delta.$$

As b^i is weakly decreasing, we can bound the left-hand side integral from below by $\frac{1}{2}\delta\left(b^i\left(q'+\frac{1}{2}\delta\right) - b^i(q'+$ hence obtaining

$$b^i\left(q'+\frac{1}{2}\delta\right) - b^i(q'+\delta) \leq \frac{2v(0)(1-M)}{P}\left(G^i(q'+\delta)-G^i(q')\right).$$

Because the density of supply is continuous and bounded away from 0 on relevant supply levels and because bidder i receives at least fraction M of any small increase in aggregate supply above Q', there is some real $\overline{f}>0$ such that $G^i(q'+\delta)-G^i(q') < \overline{f}\delta$ for sufficiently small δ. In effect,

$$b^i\left(q'+\frac{1}{2}\delta\right) - b^i(q'+\delta) \leq \frac{2v(0)\overline{f}}{P}(1-M)\delta.$$

Because this inequality holds for all δ arbitrarily small, we may telescope it to obtain

$$\lim_{k\to\infty} b^i\left(q'+\frac{1}{2^k}\delta\right) - b^i(q'+\delta) \leq \left(\sum_{k=1,2,\ldots}\frac{1}{2^k}\right)\frac{2v(0)\overline{f}}{P}(1-M)\delta,$$

where the right-hand summation converges to 2. The claim follows from the right-continuity of b^i.[59] QED

[59]Recall that we consider the failure of Lipschitz continuity in which for any $\tilde{K}$ there are arbitrarily small $\varepsilon>0$ such that $\varphi^i(p-\varepsilon)-\varphi^i(p) > \tilde{K}\varepsilon$. The argument for the failure of Lipschitz continuity in which for any $\tilde{K}$ there are arbitrarily small $\varepsilon>0$ such that $\varphi^i(p)-\varphi^i(p+\varepsilon) > \tilde{K}\varepsilon$ needs an adjustment at this point: as mentioned above, in the latter argument we replace b^i with its left-continuous modification $\hat{b}^i$. We then bound $\lim_{k\to\infty}\hat{b}^i(q'-\delta)-\hat{b}^i\left(q'-\frac{1}{2^k}\delta\right)$ from above, and the proof proceeds with minimal further changes.

Claim 4. The bids of $j \neq i$ are steep near $q^j(Q^{\min})$. That is, there is a constant $C > 0$ such that for any $M < 1$, any sufficiently small ε, and any Q' from Claim 2 sufficiently close to $Q^{\min}$, the bids near $q_j = q^j(Q')$ satisfy

$$b^j(q_j) - b^j(q_j + \varepsilon) \geq \left[\frac{M}{1-M}\right] C\varepsilon.$$

Proof. Let $q' = q^i(Q')$, M, and δ be as in Claim 3 above and $q_j = q^j(Q') = \varphi^j(b^i(q'))$ and note that when Q' is close to $Q^{\min}$ then q' is close to $q = q^i(Q^{\min})$ and q_j is close to $q^j(Q^{\min})$. Let $\varepsilon > 0$ and, for bidder $j \neq i$, consider the deviation b^ε given by

$$b^\varepsilon(q) = \begin{cases} b^i(q') & \text{if } q \in [q_j, q_j + \varepsilon], \\ b^j(q) & \text{otherwise.} \end{cases}$$

The costs and benefits of this deviation are analogous to those calculated in the proof of Claim 1 for bidder i. As the deviation is not profitable in equilibrium, we infer that

$$\left[\int_{q_j}^{q_j+\varepsilon} b^j(q_j) - b^j(x)\,dx\right] P \geq \int_{Q^{\min}}^{Q^{\max}} \int_{q^j(Q)}^{q^{\text{new}}(Q)} v(x)\,dx dF(Q)$$

where $q^{\text{new}}(Q)$ is the allocation of j after the deviation. From Lemma 8 we know that $v(q_j) > b^j(q_j)$; since $dF(\cdot) \geq \underline{f}$, this inequality implies

$$\int_{q_j}^{q_j+\varepsilon} b^j(q_j) - b^j(x)\,dx \geq C_j \int_{Q^{\min}}^{Q^{\max}} q^{\text{new}}(Q) - q^j(Q)\,dQ.$$

for some constant $C_j > 0$ that depends on neither q_j nor ε. The left-hand side can be bounded above,

$$\int_{q_j}^{q_j+\varepsilon} b^j(q_j) - b^j(x)\,dx \leq \Big(b^j(q_j) - b^j(q_j+\varepsilon)\Big)\varepsilon.$$

By Claim 2 and market clearing, we know that $q^j(Q) \leq q_j + (1-M)(Q - Q^{\min})$ and hence $Q^{\max} - Q^{\min} \geq \varepsilon/(1-M)$. As in the analysis of Claim 1, $q^{\text{new}}(Q) = \min\{q_j + \varepsilon, q_j + Q - Q^{\min}\}$. Since $q^{\text{new}}(Q^{\max}) - q^j(Q^{\max}) = 0$, we have

$$C_j \int_{Q^{\min}}^{Q^{\max}} q^{\text{new}}(Q) - q^j(Q)\,dQ \geq C_j \int_{Q^{\min}}^{\tilde{Q}} \Big(Q - Q^{\min}\Big) M dQ + C_j \int_{\tilde{Q}}^{Q^{\perp}} \varepsilon - (1-M)\Big(Q - Q^{\min}\Big)\,dQ,$$

where $Q^{\perp}$ is such that $\varepsilon - (1-\hat{M})(Q^{\perp} - Q^{\min}) = 0$ and $\tilde{Q} = Q^{\min} + \varepsilon$; we can truncate the integration at $Q^{\perp}$ because deviation b^ε weakly increases the quantity allocated to bidder j and hence $q^{\text{new}}(Q) \geq q^j(Q)$ for all Q. The right-hand side integrals are $\int_{Q^{\min}}^{\tilde{Q}} (Q - Q^{\min})\, M dQ =$

$\frac{1}{2}M\varepsilon^2$ and

$$\int_{\tilde{Q}}^{Q^{\perp}} \varepsilon - (1-M)\left(Q - Q^{\min}\right) dQ = \frac{1}{2}\left[\varepsilon - (1-M)\left(\tilde{Q} - Q^{\min}\right)\right]\left[Q^{\perp} - Q^{\min}\right]$$
$$= \frac{1}{2}M\varepsilon\left[\frac{\varepsilon}{1-M}\right],$$

where the last equation follows from the just-above definitions of $\tilde{Q}$ and $Q^{\perp}$. Putting this all together, we have

$$C_j \int_{Q^{\min}}^{Q^{\max}} q^{\text{new}}(Q) - q^j(Q)\, dQ \geq \frac{1}{2}C_j M\varepsilon^2 + \frac{1}{2}C_j M\varepsilon\left[\frac{\varepsilon}{1-M}\right] = \frac{1}{2}C_j M\left[\frac{2-M}{1-M}\right]\varepsilon^2.$$

Thus a necessary condition for the deviation not to be profitable is

$$b^j(q_j) - b^j(q_j+\varepsilon) \geq \frac{1}{2}C_j M\left[\frac{2-M}{1-M}\right]\varepsilon.$$

Because the right-hand side is positive and $2-M>1$, the claim obtains for $C = \frac{1}{2}C_j$. QED

Knowing that the bids of opponents $j \neq i$ are steep when the bid of bidder i is flat—and in particular establishing bounds for steepness and flatness in terms of common M—permits a tighter bound on the quantity lost by a downward deviation for bidder i. Retain q_i, M, and δ as above, let $\varepsilon > 0$ and consider a deviation b^{ε},

$$b^{\varepsilon}(q) = \begin{cases} b^i(q_i) - \varepsilon & \text{if } b^i(q) \in \left[b^i(q_i) - \varepsilon, b^i(q_i)\right], \\ b^i(q) & \text{otherwise.} \end{cases}$$

The cost savings of this deviation are bounded below by $P\int_{q_i}^{\varphi^i(b^i(q_i)-\varepsilon)} b^i(q) - b^{\varepsilon}(q)dq$, where P is as in Claim 1. This bound is approximated from below by

$$P\int_{q_i}^{\varphi^i\left(b^i(q_i)-\varepsilon\right)} b^i(q) - b^{\varepsilon}(q)\, dq \geq \frac{1}{2}\left(\varphi^i\left(p - \frac{1}{2}\varepsilon\right) - \varphi^i(p)\right)P\varepsilon.$$

The gross utility sacrificed is bounded above by

$$\mu\overline{f}\int_{Q^{\min}}^{\tilde{Q}} Q - Q^{\min} dQ + \mu\overline{f}\int_{\tilde{Q}}^{Q^{\max}} \frac{2(n-1)(1-M)}{CM(2-M)}\varepsilon dQ,$$

where C is as in Claim 3. The former term is the quantity lost that results in allocation $q' = q_i$ (but would have resulted in allocation $q^i(Q) > q_i$); the lost quantity in this interval is bounded above by $Q - Q^{\min}$. The latter term is the quantity lost that results in allocation $q' > q_i$; the quantity lost in this interval is bounded above by the inverse slope of opponent bids, established above. Noting that $2 - M \geq 1$, the gross utility sacrificed is bounded by

$$\left[\left(\tilde{Q} - Q^{\min}\right)^2 + \left(\frac{1-M}{M}\right)\left(Q^{\max} - \tilde{Q}\right)(n-1)\,2C^{-1}\varepsilon\right]\mu\overline{f}$$
$$\leq \left[\left[\left(\frac{1-M}{M}\right)(n-1)\,2C^{-1}\right]^2 \varepsilon^2 + \left(\frac{1-M}{M}\right)\left(Q^{\max} - \tilde{Q}\right)(n-1)\,2C^{-1}\varepsilon\right]\mu\overline{f}.$$

Note that $Q^{\max} - \tilde{Q} \leq (\varphi^i(b^i(q_i) - \varepsilon) - q_i)/M$. Substituting through, a necessary inequality is

$$\frac{1}{2}\left(\varphi^i\left(p - \frac{1}{2}\varepsilon\right) - \varphi^i(p)\right)P$$
$$\leq \left[\left(\frac{1-M}{M}\right)(n-1)\,2C^{-1}\varepsilon + \frac{1}{M}\left(\varphi^i(p-\varepsilon) - \varphi^i(p)\right)\right]\left[\left(\frac{1-M}{M}\right)(n-1)\,2C^{-1}\right]\mu\overline{f}.$$

To economize notation we let $\hat{K} = 1 - M$ and consolidate constants into C_1 and C_2 (in which we rely on M being close to 1 and thus bound M^{-1} above by 2), thus transforming the above into

$$\varphi^i\left(p - \frac{1}{2}\varepsilon\right) - \varphi^i(p) \leq \left[C_1\hat{K}\varepsilon + \left(\varphi^i(p-\varepsilon) - \varphi^i(p)\right)C_2\right]\hat{K}.$$

This gives

$$\left(\varphi^i(p-\varepsilon) - \varphi^i(p)\right)C_2\hat{K} \geq \varphi^i\left(p - \frac{1}{2}\varepsilon\right) - \varphi^i(p) - C_1\hat{K}^2\varepsilon.$$

Because the same inequality must hold for all $\varepsilon' \in (0, \varepsilon)$, telescoping this inequality implies that for any k,

$$\left(\varphi^i(p-\varepsilon) - \varphi^i(p)\right)C_2\hat{K} \geq \left[\frac{1}{C_2\hat{K}}\right]^k\left(\varphi^i\left(p - \frac{1}{2^{k+1}}\varepsilon\right) - \varphi^i(p)\right) - \frac{1}{2^k}\left[\frac{1 - \left(2C_2\hat{K}\right)^{k+1}}{1 - 2C_2\hat{K}}\right]C_1\hat{K}^2\varepsilon.$$

Since φ^i is not Lipschitz continuous at p, for any $K > 0$ and any $k \in \mathbb{N}$ we can find $\varepsilon' > 0$ such that $\varepsilon' \leq \varepsilon/2^k$ and $\varphi^i(p - \varepsilon') - \varphi^i(p) > K\varepsilon'$. For such K and ε', let $\bar{k} = \max\{k\colon \varepsilon' < \varepsilon/2^k\}$; by construction, $\varepsilon/2 < 2^{\bar{k}}\varepsilon' \leq \varepsilon$. Substituting into the previous inequality gives

$$\left(\varphi^i\left(p - 2^k\varepsilon'\right) - \varphi^i(p)\right)C_2\hat{K} \geq \left[\frac{1}{C_2\hat{K}}\right]^{\bar{k}}K\varepsilon' - \left[\frac{1 - \left(2C_2\hat{K}\right)^{\bar{k}+1}}{1 - 2C_2\hat{K}}\right]C_1\hat{K}^2\varepsilon'$$
$$\geq \left[\frac{1}{C_2\hat{K}}\right]^{\bar{k}}K\varepsilon' - 2C_1\hat{K}^2\varepsilon' = \left[\frac{K - 2\left(C_2\hat{K}\right)^{\bar{k}}C_1\hat{K}^2}{\left(C_2\hat{K}\right)^{\bar{k}}}\right]\varepsilon'.$$

The middle inequality follows from the fact that $\hat{K}$ may be arbitrarily close to 0, thus $[1-(2C_2\hat{K})^{\bar{k}+1}]/[1-2C_2\hat{K}] \leq 2$ without loss of generality. Similarly, the right-hand term in the numerator is vanishingly small in comparison to the left-hand term (which is independent of $\bar{k}$), hence

$$\varphi^i\left(p-2^{\bar{k}}\varepsilon'\right)-\varphi^i\left(p\right) \geq \frac{1}{2}\left[\frac{K}{\left(C_2\hat{K}\right)^{\bar{k}+1}}\right]\varepsilon'.$$

Recalling that $\varepsilon/2 < 2^{\bar{k}}\varepsilon' \leq \varepsilon$, we substitute into the previous inequality to obtain

$$\varphi^i\left(p-\varepsilon\right)-\varphi^i\left(p\right) \geq \varphi^i\left(p-2^{\bar{k}}\varepsilon'\right)-\varphi^i\left(p\right) \geq \frac{K\varepsilon}{\left(2C_2\hat{K}\right)^{\bar{k}+1}}.$$

Since C_2 is constant and independent of ε, and $\hat{K}$ is arbitrarily close to zero, the fact that $\bar{k}$ may be arbitrarily large implies that $\varphi^i\left(p-\varepsilon\right)-\varphi^i\left(p\right) > K'\varepsilon$ for all $K' \in \mathbb{R}$, contradicting the fact that φ^i is bounded. It follows that φ^i must be Lipschitz continuous at p. □

Lemma 13. *Equilibrium inverse bids are continuously differentiable at all prices* $p \in (\underline{p}, \overline{p}]$.

Proof. Lemma 12 gives that equilibrium inverse bids are Lipschitz continuous. Note that G_b^i is continuous at a point if the equilibrium first-order conditions are satisfied at this point; let Z be the set of quantities at which the equilibrium first-order conditions are satisfied. Because the first-order condition is satisfied almost everywhere (Lemma 11), it follows that Z has full measure and G_b^i is continuous almost everywhere (Lemma 10). Expressed in terms of inverse bid functions, the first order condition is

$$\left(v\left(\varphi^i\left(b\right)\right)-b\right)G_b^i\left(\varphi^i\left(b\right);b\right) = 1-G^i\left(\varphi^i\left(b\right);b\right) = 1-F\left(\sum_{j=1}^{n}\varphi^j\left(b\right)\right),$$

and, because the marginal value v and all inverse bids φ^i are continuous, it follows that there exists a continuous function $\hat{G}_b^i$ that equals G_b^i on Z. Because each φ^i is monotone it is differentiable on a set Z' with full measure. Thus on $Z \cap Z'$, we have

$$\varphi_p^i\left(p\right) = \frac{1}{n-1}\sum_{j\neq i}G_b^j\left(\sum_k\varphi^k\left(p\right)\right)-\frac{n-2}{n-1}G_b^i\left(\sum_k\varphi^k\left(p\right)\right).$$

It follows that there is a function $\hat{\varphi}_p^i$, continuous on all of $(\underline{p}, \overline{p}]$, such that φ_p^i equals $\hat{\varphi}_p^i$ on $Z \cap Z'$, $\varphi_p^i = \hat{\varphi}_p^i|_{Z\cap Z'}$.

Since φ^i is Lipschitz continuous it is the integral of φ_p^i, and since $\varphi_p^i = \hat{\varphi}_p^i|_{Z\cap Z'}$, it is the case that $\varphi^i(p) = -\int_p^{\overline{p}}\hat{\varphi}_p^i\left(x\right)dx$. Since $\hat{\varphi}_p^i$ is continuous, the fundamental theorem of calculus implies $\varphi_p^i = \hat{\varphi}_p^i$, and the result is shown. $\square$

Corollary 10. *In any equilibrium of the pay-as-bid auction, for all bidders i and for all $q \in [0, \overline{Q}^R/n)$,*

$$-\left(v\left(q\right)-b\left(q\right)\right)G_b^i\left(q;b^i\right)=1-G^i\left(q;b^i\right).$$

Lemma 14. *Equilibrium bidding strategies must be symmetric in all pure strategy equilibria: $b^i = b$ for all i.*

Proof. The proof proceeds by establishing an ordering of asymmetric bid functions. We use this ordering to show that equilibrium is symmetric in the $n = 2$ bidder case, and the result from the $n = 2$ bidder case provides tools for the general analysis. Intuitively, the argument is that agents would prefer to receive a positive quantity rather than zero quantity; because, as we prove, receiving zero quantities is a necessary feature of asymmetric putative equilibria, the asymmetric bids are not best responses. Our proof relies on Lemma 12, which establishes Lipschitz continuity of equilibrium inverse bids; the fundamental theorem of calculus applies, and we have that for any internal price p, $\varphi^i(p) = \int_{\underline{p}}^{p} \varphi_p^i(x)dx$.

Note that for any agent i, $\sum_{j \neq i} \varphi_p^j(p) = Q_p(p) - \varphi_p^i(p)$. Then we can write the agent's first-order condition as

$$b^i\left(q\right)=v\left(q\right)+\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_p\left(p\right)-\varphi_p^i\left(p\right)}\right).$$

Now suppose that two agents i, j have bid functions which differ on a set of positive measure; let q be such that $b^i(q) > b^j(q)$. Then there is a price p such that $\varphi^i(p) > \varphi^j(p)$, and $v(\varphi^i(p)) < v(\varphi^j(p))$. For any such price, substituting into the agents' first-order conditions gives

$$\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_p\left(p\right)-\varphi_p^i\left(p\right)}\right)>\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_p\left(p\right)-\varphi_p^j\left(p\right)}\right).$$

As $1 - F\left(Q\left(p\right)\right) \neq 0$ (because the inequality is strict), rearrangement gives

$$\varphi_p^j\left(p\right)<\varphi_p^i\left(p\right).$$

Thus, whenever $\varphi^i(p) > \varphi^j(p)$, we have $\varphi_p^i(p) > \varphi_p^j(p)$. Recalling from Theorem 1 that bids must equal values at $q = \overline{Q}/n$, this implies that if there is any p such that $\varphi^i(p) > \varphi^j(p)$, then $\varphi^i > \varphi^j$.

Now consider the implications for the $n = 2$ bidder case, and let $j \neq i$. Assume that there is p with $\varphi^i(p) > \varphi^j(p) > 0$. Then there is some $\check{p}$ such that $\varphi^j(\check{p}) = 0$ and $\varphi^i(\check{p}) > 0$.

Basic auction logic dictates that bidder i can never outbid the maximum bid of bidder j (i.e., it must be that $b^i(0) = b^j(0)$) thus it must be that bidder i's first-order condition does not apply for initial units, and she is submitting a flat bid. That is, $b^i(q)|_{q \leq \varphi^i(\check{p})} = \check{p}$. Now let $\varepsilon, \lambda > 0$, and define a deviation $\hat{b}^{\varepsilon\lambda}$ for bidder j,

$$\hat{b}^{\varepsilon\lambda}(q) = \begin{cases} b^j(0) + \lambda & \text{if } q \leq \varepsilon, \\ b^j(q) & \text{otherwise.} \end{cases}$$

Then for all $q \in (0, \varepsilon]$, $\hat{b}^{\varepsilon\lambda}(q) > b^i(q)$, and when the realized quantity is $Q \in (0, \varepsilon]$ bidder j wins the entire supply. To bound the additional utility, we see that for small $\varepsilon > 0$ bidder j gains at least

$$\int_0^\varepsilon \left(v(x) - b^j(x)\right) dx \left(F\left(\varphi^i(\check{p})\right) - F(\varepsilon)\right).$$

There is an extra cost paid as well; to bound this cost we will assume that it is paid with probability 1, and this cost is $(b^j(0) + \lambda)\varepsilon - \int_0^\varepsilon b^j(x)dx$. The deviation $\hat{b}^{\varepsilon\lambda}$ is profitable if the ratio of benefits to costs is greater than 1, hence we look at

$$\begin{aligned} &\lim_{\lambda \searrow 0, \varepsilon \searrow 0} \frac{\int_0^\varepsilon (v(x) - b^j(x))\, dx\, (F(\varphi^i(\check{p})) - F(\varepsilon))}{(b^j(0) + \lambda)\varepsilon - \int_0^\varepsilon b^j(x)\, dx} \\ &= \lim_{\varepsilon \searrow 0} \frac{\int_0^\varepsilon (v(x) - b^j(x))\, dx\, (F(\varphi^i(\check{p})) - F(\varepsilon))}{b^j(0)\varepsilon - \int_0^\varepsilon b^j(x)\, dx}. \end{aligned}$$

The numerator and denominator both go to zero as $\varepsilon \searrow 0$; application of l'Hôpital's rule gives

$$= \lim_{\varepsilon \searrow 0} \frac{v(0) - b^j(0)}{0} = +\infty.$$

Then either the deviation to $\hat{b}^{\varepsilon\lambda}$ is profitable for bidder j (when $|b^j_q(0)| < \infty$), or bidder i may (essentially) costlessly reduce the initial flat of her bid function (when $|b^j_q(0)| = \infty$).[60]

Now consider the case of $n \geq 3$ agents. By the previous arguments we know that for small quantities submitted bid functions can be ranked (as can their inverses), and that at least two agents submit the highest possible bid function. Thus, we focus on two selected

[60]Implicit here is that $v(0) > b^j(0) = b^i(0)$, which follows from Lemma 8 but in this particular case is trivial: since bidder i is bidding flat to $\varphi^i(\check{p})$, if $v(0) = b^i(0)$ she is obtaining zero surplus on a positive measure of initial units. The bidder would rather cut their bid and lose all of these units with some probability, saving payment for higher units and *gaining* expected gross utility.

inverse bid functions, defined pointwise,

$$\varphi^H(p) \equiv \max\left\{\varphi^i(p)\right\},$$
$$\varphi^L(p) \equiv \max\left\{\varphi^i(p) : \varphi^i(p) < \varphi^H(p)\right\}.$$

For any asymmetric equilibrium, φ^L is well-defined because the analysis above shows that, unless the inverse bid functions φ^i, φ^j are the same for all p, then they are different for all p. Let $m_H \equiv \#\{i\colon \varphi^i = \varphi^H\}$ and $m_L = \#\{i\colon \varphi^i = \varphi^L\}$ be the numbers of agents submitting each bid. By the above analysis $m_H \geq 2$ and $m_L \geq 1$; additionally, $m_H + m_L \leq n$. As before, there is $\check{p}$ such that $\varphi^L(\check{p}) = 0$, $\varphi^H(\check{p}) > 0$, and $\varphi^L(p) > 0$ for all $p < \check{p}$. Corollary 9 shows that φ^H must be continuous and Lemma 13 implies that φ_p^H is continuous, hence the equilibrium first order conditions imply

$$\lim_{p \searrow \check{p}} (m_H - 1)\,\varphi_p^H(p) = \lim_{p \nearrow \check{p}} \left[(m_H - 1)\,\varphi_p^H(p) + m_L \varphi_p^L(p)\right].$$

We now show that if $\lim_{p \nearrow \check{p}} \varphi_p^L(p) = 0$, then a bidder bidding b^L has a profitable deviation. Let $\varepsilon > 0$ be small, and consider a deviation $\hat{b}^L$ from b^L such that

$$\hat{b}^L(q) = \begin{cases} b^L(\varepsilon) & \text{if } q \leq \varepsilon, \\ b^L(q) & \text{otherwise.} \end{cases}$$

The deviation $\hat{b}^L$ yields a reduction in quantity bounded above by ε, at a margin bounded above by $v(0)$. Because $\varphi_p^L < \varphi_p^H \leq 0$, the probability of reduced quantity is bounded above by $(m_H + m_L)\overline{f}\varepsilon$, where $\overline{f}$ is an upper bound for $f(\cdot)$ in a neighborhood of $m_H \varphi^H(b^L(0))$. The expected gross utility loss from the deviation $\hat{b}^L$ is therefore bounded above by $(m_H + m_L)\overline{f}v(0)\varepsilon^2$. On the other hand, the deviation $\hat{b}^L$ saves the bidder payment for all quantity realizations $q > \varepsilon$. This payment is saved with probability bounded below by some $P > 0$, and, because $\varphi^L(\check{p}) = 0$ and $\lim_{p \nearrow \check{p}} \varphi_p^L(p) = 0$, for any $C > 0$ there is sufficiently small ε such that the amount saved bounded from below by ε^2/C. The deviation is profitable if

$$(m_H + m_L)\,\overline{f}v(0)\,\varepsilon^2 < \frac{\varepsilon^2}{C}.$$

After factoring out the common ε^2 term, the left-hand side is constant while the right-hand side can be arbitrarily large for small C. It follows that $\hat{b}^L$ is a profitable deviation for some ε.

Then it cannot be the case that $\lim_{p \nearrow \check{p}} \varphi_p^L(p) = 0$. It follows that

$$\lim_{p \searrow \check{p}} \varphi_p^H(p) = \lim_{p \nearrow \check{p}} \varphi_p^H(p) + \frac{m_L}{m_H - 1} \varphi_p^L(p) < 0.$$

Intuitively, the bid function b^H is steeper below $\varphi^H(\check{p})$ than above, and there is a kink at this point. This implies a discontinuity in a bidder L's first-order condition near $q = 0$. For p close to but less than $\check{p}$, the first-order condition is

$$\begin{aligned} & -\left(v\left(\varphi^L(p)\right) - p\right) f(Q(p)) \left(m_H \varphi_p^H(p) + (m_L - 1)\varphi_p^L(p)\right) - (1 - F(Q(p))) = 0, \\ \implies \quad & -\left(v\left(\varphi^L(p)\right) - p\right) f(Q(p)) \left((m_H - 1)\varphi_p^H(p) + m_L \varphi_p^L(p)\right) - (1 - F(Q(p))) > 0. \end{aligned}$$

Letting $p \nearrow \check{p}$, we know that the term $[(m_H - 1)\varphi_p^H(p) + m_L \varphi_p^L(p)]$ approaches $\lim_{p \searrow \check{p}} (m_H - 1)\varphi_p^H(p)$, proportional to the marginal probability gained by a slight increase in bid from b^L near $\check{p}$ to $\tilde{b}^L > \check{p}$. Thus, the L bidder's second-order conditions are not satisfied near $q = 0$, and this is not an equilibrium. □

E Proofs for Section 4 (Pay-as-Bid Equilibrium)

For our proofs of Theorems 2, 3, and 4, we assume that the reserve price is $R = 0$. In this case, the maximum realizable quantity is $\overline{Q}^R = \overline{Q}$. In Supplementary Appendix E.4 we detail how these proofs must change to account for binding reserve prices.

E.1 Proof of Theorem 2 (Uniqueness)

Proof. From Lemma 11 and market clearing, we know that for all bidders

$$(p(Q) - v(q))\, G_b^i\left(q; b^i\right) = 1 - G^i\left(q; b^i\right).$$

Since Lemma 14 tells us that agents' strategies are symmetric, Lemma 10 allows us to write this as

$$\left(p(Q) - v\left(\frac{1}{n}Q\right)\right)(n-1)\,\varphi_p(p(Q)) = H(Q),$$

where $H(Q) = (1 - F(Q))/f(Q)$. From market clearing, we know that $p(Q) = b(Q/n)$; hence $p_Q(Q) = b_q(Q/n)/n$. Additionally, standard rules of inverse functions give $\varphi_p(p(Q)) = 1/b_q(Q/n)$ almost everywhere. Thus we have

$$\left(p(Q) - v\left(\frac{1}{n}Q\right)\right)\frac{n-1}{n} = H(Q)\, p_Q(Q).$$

Now suppose that there are two solutions, p and $\hat{p}$. From Theorem 1 we know that $p(\overline{Q}) = \hat{p}(\overline{Q})$. Suppose that there is a Q such that $\hat{p}(Q) > p(Q)$; taking Q near the supremum of Q for which this strict inequality obtains we conclude that $\hat{p}_Q(Q) < p_Q(Q)$.[61] But then we have

$$\hat{p}\left(Q\right) > p\left(Q\right) = v\left(\frac{1}{n}Q\right) + \left(\frac{n}{n-1}\right)H\left(Q\right)p_Q\left(Q\right) > v\left(\frac{1}{n}Q\right) + \left(\frac{n}{n-1}\right)H\left(Q\right)\hat{p}_Q\left(Q\right).$$

The presumed right-continuity of bids and Lipschitz continuity of φ (from Lemma 12) allow us to conclude that if p solves the first-order conditions, $\hat{p}$ cannot.[62] □

E.2 Proof of Theorem 3 (Bid Representation)

From the first order condition established in the proof of uniqueness, the equilibrium price satisfies

$$p_Q = p\tilde{H} - \hat{v}\tilde{H},$$

where $\hat{v}(x) = v(x/n)$, and $\tilde{H}(x) = [1/H(x)][(n-1)/n]$. The solution to this equation has general form

$$p\left(Q\right) = Ce^{\int_0^Q \tilde{H}(x)dx} - e^{\int_0^Q \tilde{H}(x)dx}\int_0^Q e^{-\int_0^x \tilde{H}(y)dy}\tilde{H}\left(x\right)\hat{v}\left(x\right)dx,$$

parametrized by $C \in \mathbb{R}$. Define $\rho = \frac{n-1}{n} \in [\frac{1}{2}, 1)$. We can see that $\tilde{H} = -\rho\frac{d}{dQ}\ln(1-F)$. Thus we have

$$e^{\int_0^t \tilde{H}(x)dx} = e^{-\rho\int_0^t \frac{d}{dx}\ln(1-F(x))dx} = e^{-\rho(\ln(1-F(t))-\ln 1)} = \left(1-F\left(t\right)\right)^{-\rho}.$$

Substituting and canceling, we have for $Q < \overline{Q}$:

$$p\left(Q\right) = \left(C - \rho\int_0^Q f\left(x\right)\left(1-F\left(x\right)\right)^{\rho-1}\hat{v}\left(x\right)dx\right)\left(1-F\left(Q\right)\right)^{-\rho}. \tag{7}$$

[61] The inequality inversion here from usual derivative-based approaches reflects the fact that we are "working backward" from $\overline{Q}$, while any solution must be weakly decreasing: thus a small *reduction* in Q should yield $\hat{p}(\overline{Q}) = p(\overline{Q}) \leq p < \hat{p}$.

[62] The first-order condition for bids ensures that the slope of φ is strictly negative; then since φ is Lipschitz continuous (by Lemma 12) any equilibrium inverse bid is the integral of its own derivative, and any equilibrium market price function is the integral of its own derivative.

Since $1-F(\overline{Q})=0$, this implies that $C=\rho\int_0^Q f(x)(1-F(x))^{\rho-1}\hat{v}(x)\,dx$. The market clearing price is then given by

$$p(Q)=\rho\int_Q^{\overline{Q}} f(x)(1-F(x))^{\rho-1}\hat{v}(x)\,dx\ \ (1-F(Q))^{-\rho}.$$

Since $d/dy[F^{Q,n}(y)]=\rho f(y)(1-F(y))^{\rho-1}(1-F(Q))^{-\rho}$, our formula for market price obtains, and since we have proven earlier that the equilibrium bids are symmetric, the formula for bids obtains as well.

E.3 Proofs of Theorem 4 (Existence) and Corollary 1

Proof of Theorem 4. The proof of equilibrium existence under deterministic supply is given in the main text, therefore we assume in this proof that supply has full support, $\operatorname{Supp} Q=[0,\overline{Q}]$. Let us this fix a bidder i whose incentives we will analyze, and assume that other bidders $j\neq i$ follow the strategies $b^j=b$ of Theorem 3 when bidding on quantities $q\leq\overline{Q}^R/n$, and that they bid $b^j\left(\overline{Q}^R/n\right)=v\left(\overline{Q}^R/n\right)$ for quantities $q\in\left[\overline{Q}^R/n,\overline{Q}^R/(n-1)\right]$ if $\overline{Q}^R=\overline{Q}$ (non-binding reserve price), and that they bid $v(q)$ for quantities $q\in\left[\overline{Q}^R/n,\overline{Q}^R/(n-1)\right]$ if $\overline{Q}^R<\overline{Q}$ (binding reserve price). The resulting bid function is valid because, by definition, b satisfies $b^j\left(\overline{Q}^R/n\right)=v\left(\overline{Q}^R/n\right)$. Note that in equilibrium there is no incentive for bidder i to lower or raise their bid on any quantity $q\geq\overline{Q}^R/n$ and we only need to check that bidder i finds it optimal to submit bids prescribed by Theorem 3 for quantities $q\in[0,\overline{Q}^R/n)$.

Because the bid b derived in Theorem 3 is strictly decreasing on $[0,\overline{Q}^R/n]$ and the auction is discriminatory, a bid $\tilde{b}$ such that there is a q with $\tilde{b}(q)>b(0)$ is weakly dominated by a bid which is never above $b(0)$. Second, since the maximum of reserve price and opponents' bid b is never below $v(\overline{Q}^R/n)$ on $[0,\overline{Q}^R/(n-1)]$, a bid $\tilde{b}(q)<v(\overline{Q}^R/n)$ is never awarded quantity q. These two facts in turn imply that the bidder's optimal bid for any quantity is $\tilde{b}(q)\in[v(\overline{Q}^R/n),b(0)]$. Finally, since bid b is continuous and, by Theorem 1, is such that $b(\overline{Q}^R/n)=v(\overline{Q}^R/n)$, it is the case that for any utility-maximizing bid $\tilde{b}$ and any quantity q, there is a quantity $\hat{q}\in[0,\overline{Q}^R/n]$ such that $\tilde{b}(q)=b(\hat{q})$. Because b is strictly decreasing on $\hat{q}\in[0,\overline{Q}^R/n]$, the preceding equality defines a unique mapping $\tilde{q}$ from q to $\hat{q}$. As shown in the proof of Lemma 11, bidder i's expected utility from submitting bid $\tilde{b}$ is[63]

$$\mathbb{E}\left[u^i\left(\tilde{b}\right)\right]=\int_0^{\overline{Q}}\left(v(q)-\tilde{b}(q)\right)\left(1-F\left(q+(n-1)\varphi\circ\tilde{b}(q)\right)\right)dq,$$

[63] When $\tilde{b}(q)=b(q)$ then $\varphi\circ\tilde{b}(q)=q$. Because $1-F(nq)=0$ for $q>\frac{\overline{Q}}{n}$, we can write the utility as $\mathbb{E}\left[u^i(b)\right]=\int_0^{\overline{Q}/n}(v(q)-b(q))(1-F(nq))\,dq$. Because $b(q)=v(q)$ for $q\in\left[\frac{\overline{Q}^R}{n},\frac{\overline{Q}}{n}\right]$, we can simplify the utility further to $\mathbb{E}\left[u^i(b)\right]=\int_0^{\overline{Q}^R/n}(v(q)-b(q))(1-F(nq))\,dq$.

and it follows that we may write the expected utility from bidding $b \circ \tilde{q}$ as

$$\mathbb{E}\left[u^i\left(b \circ \tilde{q}\right)\right] = \int_0^{\overline{Q}} \left(v\left(q\right) - b \circ \tilde{q}\left(q\right)\right)\left(1 - F\left(q + \left(n-1\right)\tilde{q}\left(q\right)\right)\right) dq = \int_0^{\overline{Q}} U\left(\tilde{q}\left(q\right); q\right) dq.$$

In particular, instead of bidder i selecting a bid for quantity q, we may consider bidder i as selecting a bid such that their opponents each receive quantity $\tilde{q}(q)$.

From $U(\hat{q}(q); q) \leq \max_{\hat{q} \in [0, \overline{Q}^R/n]} U(\hat{q}; q)$, we then infer that

$$\mathbb{E}\left[u^i\left(\tilde{q}\right)\right] \leq \int_0^{\overline{Q}} \max_{\hat{q} \in \left[0, \frac{1}{n}\overline{Q}^R\right]} U\left(\tilde{q}; q\right) dq.$$

In particular, any bid which maximizes $U(\cdot; q)$ pointwise for almost every quantity q will maximize the bidder's expected utility. As we showed in Appendix D.2, the first derivative of $U(\cdot; q)$ is the pointwise first-order condition used to derive the bid b, and is equal to zero at $\tilde{q} = q$. Then by the assumption of this theorem, $U(\cdot; q)$ is maximized at $\tilde{q} = q$ for almost every q, and thus $\hat{b} = b$ is a best response to bidder i's opponents submitting the symmetric bid $b^j = b$. □

Proof of Corollary 1. Denote by φ^n the equilibrium inverse bid when there are n bidders. Note that for every $q \in [0, \overline{Q}^R/n)$ and $p \in (v(\overline{Q}^R/n), v(q))$, the expression

$$\left(v\left(q\right) - p\right)\left(1 - F\left(q + \left(n-1\right)\varphi^n\left(p\right)\right)\right)$$

is differentiable in p, nonnegative, and has limit 0 as $p \to v(q)$. To establish the condition in Theorem 4, it is thus sufficient to show that, for almost all relevant q, the derivative of this expression with respect to p is zero at most once.

The derivative is

$$-\left(1 - F\left(q + \left(n-1\right)\varphi^n\left(p\right)\right)\right) - \left(v\left(q\right) - p\right)\left(n-1\right) f\left(q + \left(n-1\right)\varphi^n\left(p\right)\right)\varphi_p^n\left(p\right). \quad (8)$$

From the equilibrium derivation in Theorem 3, this derivative is zero at $p = b^n(q)$. We now show that when n is large this derivative is negative for $p > b^n(q)$ and positive for $p < b^n(q)$.

Our first step is to show that, under the assumptions of the Corollary the slope of the inverse bid, φ_p^n, is bounded and bounded away from zero. Because $\varphi_p^n(p) = 1/b_q^n(\varphi^n(p))$, it is sufficient to show that the slope of the equilibrium bid, b_q^n, is bounded and bounded away from zero. Integrating our bid representation (1) by parts gives

$$b^n(q) = v(q) + \int_q^{\frac{\overline{Q}^R}{n}} v_q(x)\left(1 - F^{nq,n}(x)\right) dx.$$

The right-hand expression can be rewritten in terms of per capita supply, giving

$$b^n(q) = v(q) + \int_q^{\overline{Q}^{R,\text{per capita}}} v_q(x)\left(\frac{1 - F^{\text{per capita}}(x)}{1 - F^{\text{per capita}}(q)}\right)^{\frac{n-1}{n}} dx.$$

Then the derivative of the equilibrium bid function is

$$b_q^n(q) = \frac{n-1}{n}\int_q^{\overline{Q}^{R,\text{per capita}}} v_q(x)\left(\frac{1 - F^{\text{per capita}}(x)}{1 - F^{\text{per capita}}(q)}\right)^{\frac{n-1}{n}} \frac{f^{\text{per capita}}(q)}{1 - F^{\text{per capita}}(q)} dx.$$

We first show that b_q^n is bounded away from zero. Recalling that $b_q^n \leq 0$, that $\overline{v} \leq v_q(x) \leq \underline{v} < 0$ by assumption, and that $0 < \underline{f}^{\text{per capita}} < f^{\text{per capita}} < \overline{f}^{\text{per capita}}$ by assumption, we have

$$\begin{aligned}
b_q^n(q) &\leq \frac{n-1}{n}\left(\frac{1}{1 - F^{\text{per capita}}(q)}\right)^{\frac{n-1}{n}+1} \underline{v}\underline{f}^{\text{per capita}} \int_q^{\overline{Q}^{R,\text{per capita}}} \left(1 - F^{\text{per capita}}(x)\right)^{\frac{n-1}{n}} dx \\
&\leq \frac{n-1}{n}\left(\frac{1}{1 - F^{\text{per capita}}(q)}\right)^{\frac{n-1}{n}+1} \underline{v}\underline{f}^{\text{per capita}} \int_q^{\overline{Q}^{R,\text{per capita}}} \left(1 - F^{\text{per capita}}(x)\right)^{\frac{n-1}{n}} \frac{f^{\text{per capita}}(x)}{\overline{f}^{\text{per capita}}} dx \\
&\leq \frac{n-1}{n}\left(\frac{1}{\frac{n-1}{n}+1}\right)\frac{\underline{v}\underline{f}^{\text{per capita}}}{\overline{f}^{\text{per capita}}} = \frac{n-1}{2n-1}\left[\frac{\underline{v}\underline{f}^{\text{per capita}}}{\overline{f}^{\text{per capita}}}\right] \leq \frac{1}{3}\left[\frac{\underline{v}\underline{f}^{\text{per capita}}}{\overline{f}^{\text{per capita}}}\right] < 0.
\end{aligned}$$

To see that b_q^n is bounded below follows a similar path,

$$\begin{aligned}
b_q^n(q) &\geq \left(\frac{\overline{f}^{\text{per capita}}}{\left(\overline{Q}^{R,\text{per capita}} - q\right)\underline{f}^{\text{per capita}}}\right) \int_q^{\overline{Q}^{R,\text{per capita}}} \overline{v} dx \\
&= \left(\frac{\overline{f}^{\text{per capita}}}{\left(\overline{Q}^{\text{per capita}} - q\right)\underline{f}^{\text{per capita}}}\right)\left(\overline{Q}^{R,\text{per capita}} - q\right)\overline{v} = \frac{\overline{f}^{\text{per capita}}\overline{v}}{\underline{f}^{\text{per capita}}}.
\end{aligned}$$

Then b_q^n, and hence φ_p^n, is bounded and bounded away from zero. Note that these bounds are independent of the number of bidders n.

Because the density $f^{\text{per capita}}$ and its derivative $f_q^{\text{per capita}}$ are bounded, and because φ_p^n is bounded uniformly for all n, we can write (8) as

$$
\begin{aligned}
&-\left(1-F^{\text{per capita}}\left(\frac{q+(n-1)\varphi^n(p)}{n}\right)\right)-\frac{n-1}{n}(v(q)-p)\,f^{\text{per capita}}\left(\frac{q+(n-1)\varphi^n(p)}{n}\right)\varphi^n(p)\\
&=-\left(1-F^{\text{per capita}}(\varphi^n(p))\right)-\frac{n-1}{n}(v(q)-p)\,f^{\text{per capita}}(\varphi^n(p))\,\varphi^n_p(p)\\
&\quad-\left(F^{\text{per capita}}(\varphi^n(p))-F^{\text{per capita}}\left(\frac{q+(n-1)\varphi^n(p)}{n}\right)\right)\\
&\quad-\frac{n-1}{n}(v(q)-p)\left(f^{\text{per capita}}\left(\frac{q+(n-1)\varphi^n(p)}{n}\right)-f^{\text{per capita}}(\varphi^n(p))\right)\varphi^n_p(p)\\
&=-\left(1-F^{\text{per capita}}(\varphi^n(p))\right)-\frac{n-1}{n}(v(q)-p)\,f^{\text{per capita}}(\varphi^n(p))\,\varphi^n_p(p)-\frac{1}{n}(q-\varphi^n(p))\,\hat{C}_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{1}{n}(q-\varphi^n(p))\,\hat{C}_1&=-\left(F^{\text{per capita}}(\varphi^n(p))-F^{\text{per capita}}\left(\frac{q+(n-1)\varphi^n(p)}{n}\right)\right)\\
&\quad-\frac{n-1}{n}(v(q)-p)\left(f^{\text{per capita}}\left(\frac{q+(n-1)\varphi^n(p)}{n}\right)-f^{\text{per capita}}(\varphi^n(p))\right)\varphi^n_p(p)\\
&=\frac{1}{n}(q-\varphi^n(p))\,c_{F^{\text{per capita}}}-\frac{1}{n}\left[\frac{n-1}{n}(v(q)-p)\,\varphi^n_p(p)\right](q-\varphi^n(p))\,c_{f^{\text{per capita}}}\\
&=\frac{1}{n}(q-\varphi^n(p))\,c_{F^{\text{per capita}}}-\frac{1}{n}(q-\varphi^n(p))\,c_\delta c_{f^{\text{per capita}}}\\
&=\frac{1}{n}(q-\varphi^n(p))\left(c_{F^{\text{per capita}}}-c_\delta c_{f^{\text{per capita}}}\right).
\end{aligned}
$$

The constants $c_{F^{\text{per capita}}}$ and $c_{f^{\text{per capita}}}$ exist and are bounded, independent of p, q, and n, because $f^{\text{per capita}}$ and $f^{\text{per capita}}_q$ are bounded. The constant c_δ is bounded, independent of p, q, and n, because $v(q)-p$ and $\varphi^n_p(p)$ are bounded, independent of n. It follows that the constant $\hat{C}_1$ exists and has a uniform bound which is independent of p, q, and n. From our equilibrium bid representation, we may then write (8) as

$$
\frac{n-1}{n}(v(\varphi^n(p))-p)\,\varphi^n_p(p)-\frac{n-1}{n}(v(q)-p)\,\varphi^n_p(p)-\frac{1}{n}(q-\varphi^n(p))\,\frac{\hat{C}_1}{f^{\text{per capita}}(\varphi^n(p))}.
$$

Since $f^{\text{per capita}}$ and φ^n_p are bounded away from zero, (8) has the same sign as

$$
-\left[(v(\varphi^n(p))-v(q))-\frac{1}{n}(q-\varphi^n(p))\,\hat{C}_2\right],
$$

where $\hat{C}_2=\hat{C}_1/\varphi^n_p(p)$ is bounded because φ^n_p is bounded away from 0 uniformly for all n. Further, because the derivative of v is bounded away from zero, there is $\gamma<0$ such that the derivative we study has the same sign as

$$-\left[\left(\varphi^{n}\left(p\right)-q\right)\gamma-\frac{1}{n}\left(q-\varphi^{n}\left(p\right)\right)\hat{C}_{2}\right]=\left(\varphi^{n}\left(p\right)-q\right)\left(\left|\gamma\right|-\frac{1}{n}\hat{C}_{2}\right).$$

Although the specific values of γ and $\hat{C}_2$ depend on p, q, and n, they are nonetheless uniformly bounded. Since γ is bounded away from zero, it follows that there is n sufficiently large so that (8) is negative when $p > b^n(q)$ and positive when $p < b^n(q)$, completing the proof.

$\square$

E.4 Modifying the Proofs to Allow for Reserve Prices

The bound on market price established in Theorem 1 implies that a binding reserve price is equivalent to creating an atom in the supply distribution at the quantity at which marginal value equals the reserve price. In order to extend the previous proofs to the setting that allows reserve prices (as the results are stated in the main text), we therefore need to extend them to distributions in which there might be an atom at the upper bound of support $\overline{Q}$.[64] All our results remain true, and the proofs go through without much change except for the end of the proof of Theorem 3, where more care is needed.

The proof of Theorem 3 goes through until the claim that $1 - F(\overline{Q}) = 0$; in the presence of an atom at $\overline{Q}$ this claim is no longer valid. We thus proceed as follows. We multiply both sides of equation (7) by $(1 - F(Q))^{\rho}$ and conclude that

$$p\left(Q\right)\left(1-F\left(Q\right)\right)^{\rho}=C-\rho\int_{0}^{Q}f\left(x\right)\left(1-F\left(x\right)\right)^{\rho-1}v\left(\frac{1}{n}x\right)dx.$$

Now, let $\overrightarrow{F}(\overline{Q}) \equiv \lim_{Q' \nearrow \overline{Q}} F(Q')$. Because the market price and the right-hand integral are continuous as $Q \nearrow \overline{Q}$, we have

$$p\left(\overline{Q}\right)\left(1-\overrightarrow{F}\left(\overline{Q}\right)\right)=C-\rho\int_{0}^{\overline{Q}}f\left(x\right)\left(1-F\left(x\right)\right)^{\rho-1}v\left(\frac{1}{n}x\right)dx.$$

The parameter C is determined by this equation. The market price function is then

$$p\left(Q\right)=\left(\frac{1-\overrightarrow{F}\left(\overline{Q}\right)}{1-F\left(Q\right)}\right)^{\rho}p\left(\overline{Q}\right)+\rho\int_{Q}^{\overline{Q}}f\left(x\right)\left(1-F\left(x\right)\right)^{\rho-1}v\left(\frac{1}{n}x\right)dx\ \left(1-F\left(Q\right)\right)^{-\rho}. \tag{9}$$

[64] Starting with a given supply distribution F with support $[0, \overline{Q}]$ and moving all probability from $[\overline{Q}^R, \overline{Q}]$ to an atom at $\overline{Q}^R$ results in a new distribution $\tilde{F}$ with support $[0, \overline{Q}^R]$, with an atom at $\overline{Q}^R$. All results apply to this new distribution, thus it is without loss of generality to assume that the mass point is at $\overline{Q}$.

Recall from Theorem 1 that $p(\overline{Q}) = v(\overline{Q}/n)$. Extending our notation to the auxiliary distribution $F^{Q,n}$, we also have

$$F^{Q,n}(\overline{Q}) - \overrightarrow{F}^{Q,n}(\overline{Q}) = 1 - \overrightarrow{F}^{Q,n}(\overline{Q}) = \left(\frac{1-\overrightarrow{F}\left(\overline{Q}\right)}{1-F\left(Q\right)}\right)^{\rho}.$$

Since $d/dy[F^{Q,n}(y)] = \rho f(y)(1-F(y))^{\rho-1}(1-F(Q))^{-\rho}$ for all $Q, y < \overline{Q}$, we have

$$\begin{aligned} p\left(Q\right) &= \left(F^{Q,n}(\overline{Q}) - \overrightarrow{F}^{Q,n}\left(\overline{Q}\right)\right) v\left(\frac{1}{n}\overline{Q}\right) + \int_{Q}^{\overline{Q}} v\left(\frac{1}{n}x\right) \frac{d}{dy}\left[F^{Q,n}\left(y\right)\right]_{y=x} dx \\ &= \int_{Q}^{\overline{Q}} v\left(\frac{1}{n}x\right) dF^{Q,n}\left(x\right), \end{aligned}$$

proving our formula for equilibrium stop-out price in the presence of an atom at $\overline{Q}$. Noting that $\overline{Q}^{R} < \overline{Q}$ implies an atom in the realized allocation distribution at $\overline{Q}^{R}$, equation 2 in Theorem 3 follows. Since equilibrium is symmetric, equation 1 is an immediate corollary. □

F Proofs for Section 5 (Designing Pay-as-Bid Auctions): Proof of Theorem 5

Theorem 5 shows that, when the designer is constrained to a reserve price R and a distribution over supply F, the optimal mechanism is deterministic. This result is distinct, and does not follow, from the analysis in Appendix A, which shows that (under regularity conditions on demand) a seller who can implement stochastic elastic supply prefers to implement a deterministic elastic supply curve. In general, fixed supply $Q^{\star}$ and reserve $R^{\star}$ is insufficiently elastic to obtain monopoly rents from all bidder signals s, and a seller who can implement an elastic supply curve will strictly prefer to do so.

Proof of Theorem 5. Consider a pure-strategy equilibrium in a pay-as-bid auction with reserve price R and supply distribution F. In Section 4 we proved that the equilibrium is essentially unique and symmetric. Furthermore, in equilibrium, for any relevant quantity q, each bidder's bid equals the resulting market-clearing price when quantity $Q = nq$ is sold; we denote this market clearing price $p(Q; R, s)$, suppressing in the notation the price's dependence on F as it is constant. We denote the resulting equilibrium revenue by $\pi(Q; R, s)$.

The seller maximizes the expected revenue $\mathbb{E}_s\left[\pi^F\right] = \mathbb{E}_s[\int_0^{\overline{Q}} \pi\left(Q; R, s\right) dF\left(Q\right)]$, where π^F denotes the seller's profits when bidders bid against distribution of supply F. When bidders' values are low relative to the reserve price, and the realized quantity is high, the reserve price is binding and the bidders receive only a partial allocation. Because the auction is discriminatory and $b(Q/n) = p(Q)$, expected revenue may be written as

$$\mathbb{E}_s\left[\pi^F\right] = \mathbb{E}_s\left[\int_0^{\overline{Q}}\int_0^{Q^R(y,s)} p\left(x;R,s\right)dxdF\left(y\right)\right]. \tag{10}$$

Integrating by parts gives

$$\begin{aligned}\mathbb{E}_s\left[\pi^F\right] =& \mathbb{E}_s\Bigg\{\left[-\left(1-F\left(y\right)\right)\int_0^{Q^R(y,s)} p\left(x;R,s\right)dx\right]\Bigg|_{y=0}^{\overline{Q}} \\ &+\int_0^{\overline{Q}}\left(1-F\left(y\right)\right)p\left(Q^R\left(y,s\right);s\right)dQ^R\left(y,s\right)\Bigg\},\end{aligned}$$

where the first addend is zero. Recognizing that Q is continuous in y and that $Q_y^R(y,s) = 1$ for $v(y/n;s) > R$ and $Q_y^R(y,s) = 0$ for $v(y/n;s) < R$, we can thus express expected revenue as

$$\mathbb{E}_s\left[\pi^F\right] = \mathbb{E}_s\left[\int_0^{\overline{Q}^R(s)}\left(1-F\left(y\right)\right)p\left(Q^R\left(y,s\right);s\right)dy\right].$$

Applying our Theorem 3 gives

$$\mathbb{E}_s\left[\pi^F\right] = \mathbb{E}_s\left[\int_0^{\overline{Q}^R(s)}\left(1-F\left(y\right)\right)\left[\left(1-F^{y,n}\left(\overline{Q}^R\left(s\right)\right)\right)v\left(\frac{1}{n}\overline{Q}^R\left(s\right);s\right)+\int_y^{\overline{Q}^R(s)} v\left(\frac{1}{n}x;s\right)dF^{y,n}\left(x\right)\right]dy\right], \tag{11}$$

where $F^{y,n}\left(x\right) = 1-\left(\frac{1-F(x)}{1-F(y)}\right)^{\frac{n-1}{n}}$ is the c.d.f. of the weighting distribution from the theorem.[65]

Applying integration by parts to the inner integral and substituting in for $F^{y,n}$ gives

$$\mathbb{E}_s\left[\pi^F\right] = \mathbb{E}_s\left[\int_0^{\overline{Q}^R(s)}\left(1-F\left(y\right)\right)v\left(\frac{1}{n}y;s\right)+\left(1-F\left(y\right)\right)^{\frac{1}{n}}\int_y^{\overline{Q}^R(s)}\frac{1}{n}v_q\left(\frac{1}{n}x;s\right)\left(1-F\left(x\right)\right)^{\frac{n-1}{n}}dxdy\right]. \tag{12}$$

[65] The outer integral in equation (11) is bounded to $[0, Q^*(s)]$, thus $y \leq Q^*(s)$ for all y and $F^{y,n}(Q^*(s))$ is well-defined. The left-hand addend in the integral results from the fact that, when $Q^*(s) < \overline{Q}$—that is, when signal-s bidders have low values for the maximum quantity, $\hat{v}(\overline{Q};s) < R$—there is a mass point in the resulting distribution of realized aggregate allocation at $Q^*(s)$; this same expression is seen in equation (9) in Appendix (E.4).

We may change the order of integration of the right-hand double integral to obtain

$$\begin{aligned}
&\int_0^{\overline{Q}^R(s)} (1-F(y))^{\frac{1}{n}} \int_y^{\overline{Q}^R(s)} \frac{1}{n} v_q\left(\frac{1}{n}x;s\right)(1-F(x))^{\frac{n-1}{n}}\,dxdy \\
&= \int_0^{\overline{Q}^R(s)} \int_0^x (1-F(y))^{\frac{1}{n}}\,dy \frac{1}{n} v_q\left(\frac{1}{n}x;s\right)(1-F(x))^{\frac{n-1}{n}}\,dx \\
&\le \int_0^{\overline{Q}^R(s)} \frac{1}{n} x v_q\left(\frac{1}{n}x;s\right)(1-F(x))\,dx,
\end{aligned}$$

where the inequality follows from the facts that $v_q \le 0$, and $1-F(y) \ge 1-F(x)$ for $y \le x$. Substituting y for x and plugging this bound in the above expression for expected profits, we have

$$\mathbb{E}_s\left[\pi^F\right] \le \mathbb{E}_s\left[\int_0^{\overline{Q}^R(s)} (1-F(y))\left(v\left(\frac{1}{n}y;s\right)+\frac{1}{n}yv_q\left(\frac{1}{n}y;s\right)\right)dy\right].$$

Notice that $xv_q(x/n;s)/n+v(x/n;s)=\pi_q^{\delta_x}(x;s)$, where $\pi^{\delta_x}(x;s)=xv(x/n;s)$ is the revenue from selling quantity x at price $v(x/n;s)$. Integrating by parts gives

$$\begin{aligned}
\mathbb{E}_s\left[\pi^F\right] &\le \mathbb{E}_s\left[\int_0^{\overline{Q}^R(s)} \pi_q^{\delta_x}(x;s)(1-F(x))\,dx\right] \\
&= \mathbb{E}_s\left[\pi^{\delta_{\overline{Q}^R(s)}}\left(\overline{Q}^R(s);s\right)\left(1-F\left(\overline{Q}^R(s)\right)\right)+\int_0^{\overline{Q}^R(s)} \pi^{\delta_x}(x;s)\,dF(x)\right] \\
&= \mathbb{E}_s\left[\int_0^{\overline{Q}} \pi^{\delta_{Q^R(x,s)}}\left(Q^R(x,s);s\right)dF(x)\right]. \qquad (13)
\end{aligned}$$

Thus,

$$\mathbb{E}_s\left[\pi^F\right] \;\le\; \int_0^{\overline{Q}} \mathbb{E}_s\left[\pi^{\delta_{Q^R(x;s)}}\left(Q^R(x,s);s\right)\right]dF(x).$$

Since there are no cross-terms in this integral, the right-hand side is maximized at a degenerate distribution which maximizes $\mathbb{E}_s[\pi^{\delta_{Q^R(x;s)}}(Q^R(x,s);s)]$. This is exactly the problem of choosing optimal feasible deterministic supply given the reserve price R. It follows that expected revenue is weakly dominated by expected revenue with optimal deterministic supply, hence optimal supply is deterministic. □

G Robust Selection and the Proofs for Section 6 (The Auction Design Game)

G.1 Robust and Semi-truthful Equilibria in Uniform Price

In the uniform-price auction, equilibrium bidding strategies are unique when the support of supply is sufficiently large as established by Klemperer and Meyer [1989]; for their argument to apply in our setting, it is sufficient that the support of supply contains $[0, \overline{Q}]$, where $\overline{Q} \geq \sup_s nv^{-1}(R; s)$. Because the bids in Klemperer and Meyer's equilibrium remain best responses even after the bidders learn the realization of supply, these bids remain in equilibrium for all supply distributions (assuming the reserve price is kept the same). This observation allows us to re-interpret Klemperer and Meyer's uniqueness result as a selection of an equilibrium that is robust to bidders' beliefs about the distribution of supply.

In the uniform-price auction, bids for quantities which are never marginal never affect utility, and are relevant only in ensuring that there is no profitable deviation from a particular best response bid curve. For example, when supply is deterministic bidders can coordinate on collusive-seeming equilibria, in which the market-clearing price is low, and high bids for nonmarginal units ensure it is not profitable for any bidder to increase their allocation by increasing their bid. The seller has the ability to almost-costlessly eliminate these equilibria by adding a small amount of randomness to aggregate supply, ensuring that all quantities remain potentially marginal. Robust bids are therefore focal in our equilibrium analysis of the uniform-price design game: bidders cannot credibly commit to bidding below robust bids, because the seller can introduce a small amount of randomness to induce (at worst) a robust bidding equilibrium.

Lemma 15. [Symmetric Equilibrium in Uniform Price] *For all signals s and any price $p(s) \in [R, v(\overline{Q}^R(s)/n; s)]$, there is a symmetric equilibrium of the uniform-price auction in which all bidders bid*

$$b(q; s) = v(q; s) + \int_q^{\frac{1}{n}\overline{Q}^R(s)} \left(\frac{q}{x}\right)^{n-1} v_q(x; s)\, dx - \left(\frac{q}{\frac{1}{n}\overline{Q}^R(s)}\right)^{n-1} \left(v\left(\frac{1}{n}\overline{Q}^R(s); s\right) - p(s)\right).$$

Proof. We follow the approach of Klemperer and Meyer [1989]: they show that there is continuum of asymmetric equilibria in uniform price, and we leverage their analysis to show that all prices given above can be supported in symmetric equilibria. First note that the above bid function b is decreasing, $b(q) \leq v(q)$ at each quantity $q \in [0, \overline{Q}^R(s)/n]$, and at the maximum quantity $\overline{Q}^R(s)/n$ bid $b\left(\overline{Q}^R(s)/n\right) \in [R, v(\overline{Q}^R(s)/n; s)]$; in particular the bids on quantities $q \in [0, \overline{Q}^R(s)/n]$ are above the reserve price. In the uniform-price auction, the first-order conditions on the inverse bid φ are

$$(v\left(Q-(n-1)\varphi\left(p\right);s\right)-p)+\left(\frac{Q-(n-1)\varphi\left(p\right)}{n-1}\right)\frac{1}{\varphi_p\left(p\right)}=0,\ \forall Q. \tag{14}$$

Woodward [2021] shows that the symmetric solution to (14) is

$$\begin{aligned} b\left(q;s\right) &= v\left(q;s\right)+\int_q^{\frac{1}{n}\overline{Q}^R(s)} \exp\left(-\left(n-1\right)\int_q^x \frac{dz}{z}\right) v_q\left(x;s\right)dx - C\exp\left(-\left(n-1\right)\int_q^{\frac{1}{n}\overline{Q}^R(s)}\frac{dx}{x}\right) \\ &= v\left(q;s\right)+\int_q^{\frac{1}{n}\overline{Q}^R(s)}\left(\frac{q}{x}\right)^{n-1} v_q\left(x;s\right)dx - \left(\frac{q}{\frac{1}{n}\overline{Q}^R\left(s\right)}\right)^{n-1} C, \end{aligned}$$

where C is a parameter that can be set so that $b(\overline{Q}^R(s)/n;s) = p(s)$. Thus to show that $b(\cdot;s)$ is an equilibrium bidding function, it is sufficient to show that the left-hand side of (14) is negative for $p' > p$ and positive for $p' < p$; equivalently, since the equation is solved at $\varphi(p) = Q/n$, for the latter point to hold it is sufficient to show that the above expression is negative for $Q > n\varphi(p')$ and positive for $Q < n\varphi(p')$. We thus check

$$\begin{aligned} &\operatorname{sign}\left[\left(v\left(Q-(n-1)\varphi\left(p'\right);s\right)-p'\right)+\frac{Q-(n-1)\varphi\left(p'\right)}{(n-1)\varphi_p\left(p'\right)}\right] \\ &= \operatorname{sign}\left[\left(\left(v\left(Q-(n-1)\varphi\left(p'\right);s\right)-p'\right)+\frac{Q-(n-1)\varphi\left(p'\right)}{(n-1)\varphi_p\left(p'\right)}\right)-\underbrace{\left(\left(v\left(n\varphi\left(p'\right)-(n-1)\varphi\left(p'\right);s\right)-p'\right)+\frac{n \cdots}{\cdots}\right.}_{=0}\right. \\ &= \operatorname{sign}\left[\left(v\left(Q-(n-1)\varphi\left(p';s\right)\right)-v\left(n\varphi\left(p'\right)-(n-1)\varphi\left(p'\right);s\right)\right)+\frac{Q-n\varphi\left(p'\right)}{(n-1)\varphi_p\left(p'\right)}\right]. \end{aligned}$$

Recalling that $\varphi_p < 0$, when $Q < n\varphi(p')$ the leading and trailing expressions are positive, and when $Q > n\varphi(p')$ the leading and trailing expressions are negative, as desired. □

The existence of semi-truthful and robust equilibria is an immediate consequence of Lemma 15. Proposition 1 gives the explicit form of robust equilibrium bids.

Proposition 1. [Bids Robust to Uncertainty] *The unique uniform-price equilibrium bid profile that is robust to uncertainty is given by*

$$b\left(q;s\right)=\left(\frac{q}{v^{-1}\left(R;s\right)}\right)^{n-1}R+(n-1)\int_q^{v^{-1}(R;s)}\left(\frac{q}{x}\right)^{n-1}\frac{v\left(x;s\right)}{x}dx, \tag{15}$$

or, equivalently,

$$b(q;s) = v(q;s) + \int_q^{v^{-1}(R;s)} \left(\frac{q}{x}\right)^{n-1} v_q(x;s)\,dx.$$

Proof. With unbounded supply, expression (15) gives the unique solution to the equilibrium first-order conditions in the uniform-price auction (Lemma 15). Then $(b^i)_{i=1}^n$ is the unique robust uniform-price bid profile. $\square$

We henceforth refer to the above uniform-price bid function as the *robust uniform-price bid*. The robust uniform-price bid is continuous, differentiable, strictly below marginal values for all $q \in (0, v^{-1}(R;s))$, and equal to marginal values for $q \in \{0, v^{-1}(R;s)\}$. No matter which auction format is employed, optimal supply $Q^\star$ is strictly positive. In the pay-as-bid design game the optimal deterministic quantity must be binding for some bidder types, $Q^{\star\mathrm{PAB}} < \sup_s v^{-1}(R;s)$, provided the value space is rich. Since robust uniform-price bids are strictly below value on $(0, Q^{\star\mathrm{PAB}}/n]$ for all types s such that $Q^{\star\mathrm{PAB}} < v^{-1}(R;s)$, the pay-as-bid auction generates strictly greater revenue than the uniform-price auction with robust bidding. Because, in the auction design game, bidders can select an equilibrium on the basis of the supply and reserve chosen by the auctioneer, revenue dominance of deterministic pay-as-bid is sufficient to prove Lemma 1.

Proof of Lemma 1. We first show that, holding bids fixed, optimal supply is deterministic in the uniform-price auction. Given bid b and distribution of per-capita supply F^μ, the expected revenue obtained from a given bidder in the uniform-price auction is

$$\begin{aligned}
&\mathbb{E}_s\left[\left(1 - F^\mu\left(\overline{Q}^R(s)\right)\right) R\overline{Q}^R(s) + \int_0^{\overline{Q}^R(s)} qb(q;s)\,dF^\mu(q)\right] \\
&= \mathbb{E}_s\left[\int_0^{\overline{Q}^R(s)} \left(b(q;s) + qb_q(q;s)\right)\left(1 - F^\mu(q)\right) dq\right] \\
&= \int_0^{\overline{Q}} \mathbb{E}_s\left[b(q;s) + qb_q(q;s) \middle| \overline{Q}^R(s) > q\right]\left(1 - F^\mu(q)\right) dq = \int_0^{\overline{Q}} J(q;s)\left(1 - F^\mu(q)\right) dq.
\end{aligned}$$

It follows that the optimal distribution F^μ is deterministic, and is equal to 0 below some threshold and 1 above it.

Following Proposition 1, robust uniform-price bids can be represented as

$$b(q;s) = v(q;s) + \int_q^{\hat{Q}(s)} \left(\frac{q}{x}\right)^{n-1} v_q(x;s)\,dx.$$

Because $v_q < 0$, these bids are strictly below values at all $q < \hat{Q}(s)$. And because optimal supply (holding bids fixed) is deterministic, optimal revenue under robust bids is strictly below optimal pay-as-bid revenue: otherwise there is a reserve R and deterministic quantity Q that yield expected uniform-price revenue equal to expected pay-as-bid revenue, contradicting the richness of the value space.

Since the maximum expected revenue obtained under robust bids is strictly below the optimal expected revenue in the pay-as-bid auction, it is sufficient to show that when $\varepsilon > 0$ is small and random supply is supported on $[Q^{\star\text{PAB}} - \varepsilon, Q^{\star\text{PAB}} + \varepsilon]$ and the reserve price is $R \in [R^{\star\text{PAB}} - \varepsilon, R^{\star\text{PAB}} + \varepsilon]$, equilibrium uniform-price revenue under semi-truthful bids is close to optimal pay-as-bid revenue. A lower bound on this revenue is

$$\mathbb{E}_s\left[\int_{Q^{\star\text{PAB}}-\varepsilon}^{Q^{\star\text{PAB}}+\varepsilon} \overline{Q}^R(Q;s)\, v\left(\frac{1}{n}\overline{Q}^R(Q;s);s\right) dF(Q)\right] \geq \mathbb{E}_s\left[\overline{Q}^R\left(\overline{Q}^{\star\text{PAB}} - \varepsilon; s\right) v\left(\frac{1}{n}\overline{Q}^R\left(\overline{Q}^{\star\text{PAB}} + \varepsilon; s\right); s\right)\right].$$

Because $\overline{Q}^R$ is continuous in Q and R, and because $R \to R^{\star\text{PAB}}$ as $\varepsilon \searrow 0$, the right-hand side converges to

$$\mathbb{E}_s\left[\overline{Q}^{R^{\star\text{PAB}}}\left(Q^{\star\text{PAB}}; s\right) v\left(\frac{1}{n}\overline{Q}^{R^{\star\text{PAB}}}\left(Q^{\star\text{PAB}}; s\right); s\right)\right].$$

This is exactly optimal pay-as-bid revenue. Then suppose that bidders play semi-truthful bids when the auctioneer selects reserve R and distribution F, and play robust bids otherwise. Provided $\varepsilon > 0$ is sufficiently small, reserve R and distribution F will yield more revenue to the auctioneer than any other selection. The result follows. □

Proposition 2. **[Strict Dominance of Pay-as-Bid Revenue]** *When the value space is rich, the pay-as-bid design game generates strictly greater revenue than the unique robust equilibrium of the uniform-price design game.*

Note that Proposition 2 depends on the richness of the value space only because, when the value space is not rich, a single quantity and reserve is optimal for all bidder signals s; when the auction's supply is given by this optimal supply and its reserve by this optimal reserve, there is a unique bidding equilibrium in both the pay-as-bid and uniform-price auctions, and equilibrium revenue will not depend on the auction format selected.

Proposition 3. **[Range of Prices in Uniform Price]** *If $p^\star(s)$ is the market-clearing price at supply $\overline{Q}^R$ in an equilibrium of the uniform-price auction with supply distribution F, then for all signals s, $p^\star(s) \in [R, v(\overline{Q}^R/n; s)]$. Furthermore, for any supply distribution, any signal s, and any $p^\star(s) \in [R, v(\overline{Q}^R/n; s)]$, there is an equilibrium of the uniform-price design game with market-clearing price at supply $\overline{Q}^R$ equal to $p^\star(s)$.*

Proof. The second claim follows from Lemma 15. To prove the first claim note that $p^\star(s) \geq R$ by definition. If $p^\star(s) > v(\overline{Q}^R/n; s) \geq R$, then with positive probability some bidder i is allocated a positive mass of q_i such that $v(q_i; s) < p^\star(s)$. If this bidder bids $b' = v(\cdot; s)$ instead, she is awarded all units she values above $p^\star(s)$ at a price no greater than $p^\star(s)$. This deviation is profitable as she keeps all positive-margin units, drops the negative-margin units, and does not increase her payment. □

G.2 Deterministic Revenue Bound in Uniform Price

Lemma 16. [**Deterministic Dominance in Uniform Price**] *For any equilibrium of the uniform-price design game* $((R, F), b)$*, there is a deterministic-supply equilibrium* $((R^\star, F^\star), b^\star(\cdot; s, R^\star, F^\star))$ *that generates weakly higher seller revenue and has the same on-path bids.*

Proof. With symmetrically-informed bidders, equilibrium bids in the uniform-price auction are optimal for every realization of supply, a point first made by Klemperer and Meyer [1989]. For a given bidder, every realization of supply determines a residual supply curve corresponding to the demands of the other bidders, and the given bidder's bid effectively serves to select the price-quantity pair from this residual supply curve; this choice does not depend on choices at other realizations of supply as long as the resulting bid curve is downward-sloping. In effect, two supply distributions with the same support admit the same set of equilibria, and if one supply distribution has a smaller support than another, its set of equilibrium bids is a weak superset of the other. This implies that the revenue-maximizing equilibrium with deterministic supply is also revenue-maximizing among all possible equilibria. □

G.3 Proof of Theorem 7

In the proof below we decorate market outcome functions with superscripts denoting the relevant mechanism, where helpful. For example, $p^{\star\mathrm{UP}}$ is the market-clearing price in the uniform-price auction and $p^{\star\mathrm{PAB}}$ is the market-clearing price in the pay-as-bid auction.

Proof of Theorem 7. As discussed in Theorem 5 and Lemma 16, we may restrict attention to optimal deterministic supply distributions in both the pay-as-bid and uniform-price auctions. Revenue maximization may then be expressed as a per-agent quantity $q^\star$ and market price $p^\star$; for signals s such that $v(q^\star; s) \geq p^\star$ it is without loss to assume that the total allocation is $nq^\star$—there is sufficient demand for the total quantity at the reserve price—while for signals s such that $v(q^\star; s) < p^\star$ it is clear that some total quantity $nq' < nq^\star$ will be allocated. The seller's expected revenue is then an expectation over bidder signals,

$$\mathbb{E}_s\left[\pi\right] = \mathbb{E}_s\left[nq\left(q^\star, p^\star; s\right) \cdot p\left(q^\star, p^\star; s\right)\right].$$

The quantity allocated under the uniform-price auction equals the quantity allocated under the pay-as-bid auction, $q^{\mathrm{UP}}(q^\star, p^\star; s) = q^{\mathrm{PAB}}(q^\star, p^\star; s)$, whenever $v(\cdot; s)$ is strictly decreasing at this quantity, or when $v(\cdot; s) > p^\star$ at this quantity.[66] Since we have assumed that $v(\cdot; s)$ is strictly decreasing, the quantity allocation depends only on $q^\star$ and $p^\star$ and not on the mechanism employed. Additionally, it is the case that $p^{\star\mathrm{UP}}(q^\star, p^\star; s) = p^{\star\mathrm{PAB}}(q^\star, p^\star; s)$ whenever $v(q^\star; s) < p^\star$. Let $\underline{\mathcal{S}}$ be the set of such s,

$$\underline{\mathcal{S}} = \{s' \colon v\left(q^\star; s\right) < p^\star\}.$$

Then we have

$$\mathbb{E}_s\left[\pi\right] = p^\star \Pr\left(s \in \underline{\mathcal{S}}\right) \mathbb{E}_s\left[nq\left(q^\star, p^\star; s\right) | s \in \underline{\mathcal{S}}\right] + nq^\star \Pr\left(s \notin \underline{\mathcal{S}}\right) \mathbb{E}_s\left[p\left(q^\star, p^\star; s\right) | s \notin \underline{\mathcal{S}}\right].$$

The left-hand term is independent of the mechanism employed, while the right-hand term depends on the mechanism only via the expected market-clearing price. In the pay-as-bid auction, we have seen that $p(q^\star, p^\star; s) = v(q^\star; s)$ for all $s \notin \underline{\mathcal{S}}$, while in the uniform-price auction any price $p \in [p^\star, v(q^\star; s)]$ is supportable in equilibrium. It follows that the pay-as-bid auction weakly revenue dominates the uniform-price auction, and generally will strictly do so. That the seller-optimal equilibrium of the uniform-price auction is revenue-equivalent to the unique equilibrium of the pay-as-bid auction arises from the selection of $p^{\star\mathrm{UP}}(q^\star, p^\star; s) = v(q^\star; s)$ for all $s \notin \underline{\mathcal{S}}$. □

H Proofs for Appendix A (Elastic Supply)

H.1 Proof of Theorem 8 (Uniqueness with Elastic Supply)

Proof. The analysis from the proof of Theorem 1 allows us to conclude that on the maximum unit each bidder might receive, the bidder pays her marginal value. Letting $\hat{Q}(s)$ be the aggregate quantity awarded in equilibrium under supply curve $Q^\star(s)$, it cannot be that $p^\star(\hat{Q}(s); s) > \hat{v}(\hat{Q}(s); s)$, since bids on relevant quantities are weakly below values. If, instead, $p^\star(\hat{Q}(s); s) < \hat{v}(\hat{Q}(s); s)$, the arguments from the proof of Theorem 1 apply; indeed, they are strengthened by the fact that a small increase in bid increases allocation not only by beating opponent bids, but also by increasing the market price and moving up the supply curve.

Because each bidder bids $b^\star(\hat{Q}(s)/n; s) = v(\hat{Q}(s)/n; s)$ in any equilibrium, each bidder's

[66] In the latter case there is excess demand, so all units will be allocated. In the former case all units are allocated at the reserve price; there is a possible difference in allocation when bidders' marginal values are flat over an interval of quantities at the reserve price, since bidders are indifferent between receiving and not receiving these quantities.

allocation is $\hat{Q}(s)/n$. This allocation is deterministic, conditional on the bidder-common signal s. Then any bid curve b such that $b(q) > v(\hat{Q}(s)/n; s)$ for some $q > 0$ is wasteful: it does not affect the resulting allocation, and $\int_0^{\hat{Q}(s)/n} b(q)dq > \int_0^{\hat{Q}(s)/n} b^\star(q; s)dq$. It follows that $b^\star(q) = v(\hat{Q}(s)/n; s)$ for all $q \leq \hat{Q}(s)/n$, and equilibrium bids are unique for all relevant quantities. □

H.2 Proof of Lemma 2

As we consider the special case of the seller who knows the bidders' values, we simplify notation and suppress the signal while writing value and bid functions.

H.2.1 Preliminary Definitions

Recall that we defined the supply-reserve distribution $K(Q; R)$ in Appendix A. For simplicity, we carry out the analysis under the assumption that supply-reserve distribution K is continuously differentiable. In Remark 1 we show that this assumption may be dropped.

Holding the supply-reserve distribution K fixed, fix a bidder i and consider the aggregate demand of her opponents. Allowing for mixed strategies and asymmetric and asymmetrically-informed bidders, we denote the aggregate demand of bidder i's opponents by $Q(\cdot; \xi)$, where ξ indexes the joint distribution of her opponents' potentially mixed strategies. As with supply-reserve distribution K, we assume that aggregate demand Q is continuously differentiable, and show in Remark 1 that this assumption may be dropped. Although we separately specify the supply-reserve distribution K and the mixed strategy index ξ because the former is controlled by the seller while the latter is not, a bidder's set of best responses does not depend on the source of randomness in that bidder's residual supply. Bidders' ex post utility is determined by realized quantity and payment, and thus the dependence of interim utility on the joint distribution of quantity and payment is unaffected by the introduction of a random reserve price, asymmetric information among bidders, and the possibility of mixed strategies. Thus, the bidder's first order condition is unchanged from the analysis in Lemma 11 (in Supplementary Appendix D), and the random reserve affects only the distribution of realized quantity. In the language of Lemma 11,

$$\begin{aligned} &G^i(q; b) = \mathbb{E}_\xi \left[K\left(q + Q(b; \xi); b\right)\right], \\ &\text{and } G_b^i(q; b) = \mathbb{E}_\xi \left[K_Q\left(q + Q(b; \xi); b\right) Q_p(b; \xi) + K_R\left(q + Q(b; \xi); b\right)\right]. \end{aligned} \tag{16}$$

For example, when the reserve price is fixed, $K_R = 0$ for all relevant prices, and (16) is identical to what we find in equation (10).

We aim to show that the seller can induce the same bidder behavior by implementing a random reserve without constraining supply, in which case $K_Q = 0$, and the bidder's pointwise first order condition is

$$(v(q) - b(q))\,\mathbb{E}_\xi\left[K_R\left(q + Q\left(b(q);\xi\right);b\right)\right] = \mathbb{E}_\xi\left[K\left(q + Q\left(b(q);\xi\right);b\right)\right].$$

As $K_Q = 0$ implies that K is independent of q (and thus Q is independent of ξ), we write this in terms of only the distribution of reserve prices F^R,

$$(v(\varphi(p)) - p)\,F^R_p(p) = F^R(p).$$

Thus a key simplification associated with random reserve and unconstrained supply is that the optimal bid is determined by the reserve distribution F^R and does not depend on opponent bids. Furthermore, for each quantity the optimal bid is either pointwise optimal, or this quantity is part of an interval on which the first order conditions are ironed, cf. Woodward [2016]. We capture these optimality conditions in the concept of first-order optimal bids defined as follow.

Definition 2. [First-order optimality] Given a distribution of reserve prices F^R, we say that b is *first-order optimal* with respect to F^R if:

1. If b is strictly decreasing at q, then it solves the pointwise first order condition: $(v(q) - b(q))F^R_p(b(q)) = F^R(b(q))$.
2. If b is constant in a neighborhood of q then $b(q)$ is a mass point of F^R and it solves the ironed first order condition:

$$\left(F^R(b(q)) - F^R(\underline{b})\right)(v(\overline{\varphi}(p)) - \underline{b}) = (b(q) - \underline{b})\,F^R\left(\underline{p}\right), \text{ where } \underline{b} = \lim_{q' \searrow \overline{\varphi}(p)} b(q').$$

Intuitively, the ironing conditions state that the marginal gain from slightly extending the constant interval (marginal additional quantity with probability $F^R(b(q)) - F^R(\underline{b})$) must equal the marginal cost from the same (marginal additional payment with probability $F^R(\underline{b})$). As b is weakly decreasing, any quantity q belongs to either an interval on which b is flat or to an interval on which b is strictly decreasing (and it might be an endpoint of both types of intervals simultaneously). The structure of these intervals can be complex, but there is at most a countable number of them.

Although optimal bids are first-order optimal the converse need not be true: first-order optimality only implies that a bid satisfies pointwise first order conditions where applicable, and ironing conditions elsewhere. In deriving the revenue bounds below, we assume only that the first-order conditions are satisfied, not that bids are optimal. Because any (globally) optimal bid function satisfies the first-order optimality conditions above, the bound on revenues applies to optimal bids.

Let $G^K(\cdot; b, Q)$ be the distribution of realized quantity given supply-reserve distribution K, bid function b, and potentially random residual supply Q, and let $G^R(\cdot; b)$ be the distribution of realized quantity given reserve distribution F^R and bid function b. As mentioned above, G^R does not depend on Q because, under random reserve, supply does not depend on opponent bids. Letting ξ represent randomness in residual supply (e.g., mixed strategies for a bidder's opponents)[67] we have

$$\begin{aligned}
G^R(q; b) &= 1 - F^R(b(q)), \\
\frac{d}{dq} G^R(q; b) &= -F^R_p(b(q))\, b_q(q); \\
G^K(q; b, Q) &= \mathbb{E}_\xi\left[K(q + Q(b(q); \xi), b(q))\right], \\
\frac{d}{db} G^K(q; b, Q) &= \mathbb{E}_\xi\left[K_q(q + Q(b(q); \xi))\, Q_p(b(q); \xi) + K_p(q + Q(b(q); \xi), b(q))\right], \\
\frac{d}{dq} G^K(q; b, Q) &= \frac{d}{db} G^K(q; b, Q)\, b_q(q) + \mathbb{E}_\xi\left[K_q(q + Q(b(q); \xi))\right]. \qquad (17)
\end{aligned}$$

The expected revenue from bidder i when the bidder bids b and the bid leads to quantity distribution $G^\cdot$ is given by $\pi(b; G^\cdot) = \int_0^Q \int_0^q b(x)\, dx dG^\cdot(q)$.

H.2.2 The Optimality of Random Reserve with Known Values

We begin with a bid function b which is a best response to residual supply distribution $G^i(\cdot; b)$ and construct a reserve price distribution and bidder's best response to this new distribution that raise more revenue.

Lemma 17. *Let b be a best response bid curve under residual supply distribution G^i, generated by supply-reserve distribution K and stochastic aggregate demand Q. There is a reserve distribution F^R and a first-order optimal response b^R to F^R such that $\pi\left(b^R; G^R\right) \geq \pi(b; G^i)$.*

While the bound on revenue in Lemma 17 might depend on the equilibrium selected, the subsequent analysis will show that this bound is weakly lower than the revenue in a unique equilibrium under deterministic elastic supply.

[67] In the main text we focus on pure strategies. In this analysis we allow for mixed strategies, allowing us to show that all randomness—exogenous or otherwise—is detrimental to the seller's revenue.

Proof. For clarity, we proceed under the assumption that supply-reserve distribution studied K and aggregate residual demand Q are continuously differentiable. Following the derivation of the result for smooth K and Q, we comment on extending the argument to potentially discontinuous K and Q.

We construct b^R and F^R by first constructing an auxiliary distribution G^R. As a preparatory step in the construction of G^R, recall that the discussion of the previous subsection shows that under a random reserve price that induces differentiable quantity distribution G^R, in any interval in which b is strictly decreasing b solves the pointwise first-order condition

$$-\left(v\left(q\right)-b\left(q\right)\right)F_{p}^{R}\left(b\left(q\right)\right)=F\left(b\left(q\right)\right).$$

In our construction of G^R we ensure that the pointwise first order conditions of an agent bidding b are satisfied; that is,

$$-\left(v\left(q\right)-b\left(q\right)\right)G_{q}^{R}\left(q\right)=\left(1-G^{R}\left(q\right)\right)b_{q}\left(q\right),$$

and thus

$$\frac{d}{dq}\ln\left[1-G^{R}\left(q\right)\right]=\frac{b_{q}\left(q\right)}{v\left(q\right)-b\left(q\right)}.$$

Given any initial value of $G^R(q)$ (initial condition of the ODE), we can solve this differential equation for any differentiable $b < v\left(q\right)$. In particular, for any quantity $\tilde{q}$ such that b is strictly decreasing on $(\tilde{q}, q)$, we obtain

$$G^{R}\left(\tilde{q}\right)=1-\exp\left(\int_{q}^{\tilde{q}}\frac{b_{q}\left(x\right)}{v\left(x\right)-b\left(x\right)}dx\right)\left[1-G^{R}\left(q\right)\right]. \tag{18}$$

We now construct G^R and we show that $G^R \succeq_{\text{FOSD}} G^K$; in particular, G^R puts more weight on larger quantities than G^K does. To start, let $G^R(0) = G^K(0)$. We say that an open interval $(\tilde{q}_\ell, \tilde{q}_r)$ is maximal with respect to a property if the property is satisfied on this interval but not on any other open interval containing $(\tilde{q}_\ell, \tilde{q}_r)$. At the left endpoint of any maximal interval $(\tilde{q}_\ell, \tilde{q}_r)$ on which b is strictly decreasing, we define G^R so that $G^R(\tilde{q}_\ell) = G^K(\tilde{q}_\ell)$, and we define G^R on the interior of $(\tilde{q}_\ell, \tilde{q}_r)$ so that b satisfies the first-order ODE given the initial condition $G^R(\tilde{q}_\ell)$. In particular, the first-order ODE determines the value at the right endpoint of the strictly decreasing b interval, $G^R(\tilde{q}_r)$. For any maximal open interval (q_ℓ, q_r) on which b is constant, let the value at the right endpoint be $G^R(q_r) = G^K(q_r)$. (Importantly, G^R is well defined at $q_r = \tilde{q}_\ell$ at which the right endpoint q_r of constant-b interval coincides with the left endpoint $\tilde{q}_\ell$ of strictly-decreasing-b interval). Notice that for any maximal interval (q_ℓ, q_r) on which b is constant, q_ℓ is either 0 or equal to a limit of a sequence of the right end points of maximal intervals.[68] We will see below that the values of G^R on this sequence are monotonic. Since they are also bounded below (they are nonnegative), the sequence of values of G^R at these right endpoints converges, and we define $G^R(q_\ell)$ as its limit, and also set $G^R(q) = G^R(q_\ell)$ for q in in the interior of any maximal open interval (q_ℓ, q_r) on which b is constant. This concludes the construction of G^R for all quantities strictly lower than the maximum possible quantity; at this quantity we set G^R to equal 1. Thus G^R is a c.d.f. iff it is monotone.

To establish monotonicity, suppose that q_ℓ, q_r are such that $q_\ell < q_r$, $G^R(q_\ell) \leq G^K(q_\ell)$, and that b is strictly decreasing on (q_ℓ, q_r). Then on (q_ℓ, q_r), the pointwise first-order optimality conditions obtain, and we have

$$-\left(v\left(q\right)-b\left(q\right)\right)G_q^R\left(q\right)=\left(1-G^R\left(q\right)\right)b_q\left(q\right), \text{ and } -\left(v\left(q\right)-b\left(q\right)\right)G_b^K\left(q\right)=1-G^K\left(q\right);$$

in particular, G^R and G^K are continuous on (q_ℓ, q_r). The left-hand equation holds by construction of G^R and the right-hand equation follows from the fact that b is a best response to supply-reserve distribution K and opponent demand Q. By construction, the $-(v(q)-b(q))$ terms are equal, and so for any $q \in (q_\ell, q_r)$ it must be that

$$\frac{G_q^R\left(q\right)}{1-G^R\left(q\right)}=\frac{G_b^K\left(q\right)b_q\left(q\right)}{1-G^K\left(q\right)}. \tag{19}$$

Suppose that there is $q \in (q_\ell, q_r)$ such that $G^R(q) > G^K(q)$. Then there is $\hat{q} \in (q_\ell, q)$ such that $G^R(\hat{q}) = G^K(\hat{q})$, because the c.d.fs G^R and G^K are continuous on (q_ℓ, q_r) and $G^R(q_\ell) \leq G^K(q_\ell)$. At this $\hat{q}$, equation 19 becomes $G_q^R(\hat{q}) = G_b^K(\hat{q})b_q(\hat{q})$, and substituting in for equations 17 gives

$$G_q^R\left(\hat{q}\right)=G_b^K\left(\hat{q}\right)b_q\left(\hat{q}\right)=G_q^K\left(\hat{q}\right)-\mathbb{E}_\xi\left[K_q\left(q+Q\left(b\left(q\right);\xi\right)\right)\right]\leq G_q^K\left(\hat{q}\right).$$

We conclude that $G^K(\hat{q}) = G^R(\hat{q})$ implies $G_q^K(\hat{q}) > G_q^R(\hat{q})$, contradicting $G^R(q) > G^K(q)$. From this it follows that if b is strictly decreasing on $[q_\ell, q_r]$ and $G^R(q_\ell) \leq G^K(q_r)$, then $G^R|_{q\in[q_\ell,q_r]} \succeq_{\text{FOSD}} G^K|_{q\in[q_\ell,q_r]}$, and, in particular, $G^R(q_r) \leq G^K(q_r)$. This inequality allows us to conclude that if q_r is the limit of left endpoints $\tilde{q}_\ell > q_r$ of maximal intervals, then $G^R(q_r)$ is weakly below the limit of $G\left(\tilde{q}_\ell\right)$ on this sequence. We can conclude that that G^R is monotonic and hence a cumulative distribution function such that $G^R \succeq_{\text{FOSD}} G^K$.

We now define the random reserve distribution F^R as follows: for any q, let $F^R(b(q)) = 1 - G^R(q)$. We construct a bid function b^R that is first-order optimal with respect to F^R

[68]The limit might be over right endpoints of both strictly decreasing b and constant b intervals. We allow for a constant sequence, that is the case where q_ℓ is the right end point of an adjacent interval.

and such that $b^R \geq b$. Our construction is iterative: we begin with $b^{R0} = b$, then show how to compute $b^{R[t+1]}$ from b^{Rt} for any $t \geq 0$. Let Ω_t be the set of maximal constant intervals of b^{Rt}. For an interval $(q_\ell, q_r) \in \Omega_t$, let $\tilde{q}_r$ solve the ironed first-order optimality condition at bid level $b^{Rt}(q_r)$,[69]

$$\left(F^R\left(b^{Rt}\left(q_r\right)\right) - \lim_{q \searrow q_r} F^R\left(b^{Rt}\left(q\right)\right)\right)\left(v\left(\tilde{q}_r\right) - b^{Rt}\left(q_r\right)\right) = \left(b^{Rt}\left(q_r\right) - b^{Rt}\left(\tilde{q}_r\right)\right) F^R\left(b^{Rt}\left(\tilde{q}_r\right)\right).$$

Since $p = b^{Rt}(q_r)$ is a level at which b is constant, there is a mass point in F^R at $b^{Rt}(q_r)$, and the first-order ironing equation cannot be solved at $\tilde{q}_r < q_r$. It follows that $\tilde{q}_r \geq q_r$, and moreover that $b^{Rt}(\tilde{q}_r) \leq v(\tilde{q}_r)$. Then let $\tilde{\Omega}_t$ be the set of intervals $(q_\ell, \tilde{q}_r)$, where $(q_\ell, q_r) \in \Omega_t$ and $\tilde{q}_r$ is derived from q_r as above. We now define $b^{R[t+1]}$,

$$b^{R[t+1]}\left(q\right) = \begin{cases} \sup\left\{b^{Rt}\left(q_r\right) : q \in \left(q_\ell, \tilde{q}_r\right) \in \tilde{\Omega}_t\right\} & \text{if } \exists \left(q_\ell, \tilde{q}_r\right) \in \tilde{\Omega}_t \text{ with } q \in \left(q_\ell, \tilde{q}_r\right), \\ b^{Rt}\left(q\right) & \text{otherwise.} \end{cases}$$

By construction, $b^{Rt} \leq b^{R[t+1]} \leq v$, and thus $b^{Rt} \to b^R$ for some b^R.[70] Where the limit b^R is strictly decreasing, it is equal to b and therefore satisfies the first-order conditions for optimality. When the limit b^R is constant, it satisfies the ironed first-order conditions for optimality by construction. It follows that b^R is first-order optimal. Finally, since $b = b^{R0}$ and $b^{Rt} \leq b^{R[t+1]}$ for all t, it must be that $b \leq b^R$.

Being weakly higher than b, the bid function b^R induces a realized quantity distribution $\tilde{G}^R$ that is weakly stronger than G^R (the distribution of realized quantity with reserve distribution F^R and bid b), which is in turn weakly stronger than G^K, and it follows that $\pi(b^R; \tilde{G}^R) \geq \pi(b; G^K)$. Since F^R implements b^R as a first-order optimal bid function, the lemma follows. $\square$

Remark 1. When supply-reserve distribution K and aggregate supply Q are discontinuous, we adjust the first condition of the definition of a bidder's first-order optimality at points at which G^K is not differentiable and require at these points that the left derivative with

[69]Measure-zero changes in bid do not affect utility or incentives. Therefore in this proof we assume, without loss of generality, that b^{Rt} is left continuous.

[70]Note that in the simple case where the original bid function b is strictly decreasing, it is the case that $b^R = b$. The iterative process applied here handles the possible need to extend to the right the constant intervals from the original bid function b, as well as the possibility that one constant interval "overtakes" another in the iterative process. Note that in the latter case $b^R(q) > b(q)$ for q in the overtaken constant interval of b.

respect to b (which always exists, since G^K is decreasing in b) satisfies[71]

$$-\left(v\left(q\right)-b\left(q\right)\right)G_{b-}^{i}\left(q;b\right)-\Big(1-G^{K}\left(q;b\right)\Big)\geq 0.$$

This is the only adjustment in the definition; the previous definition is unchanged at points of differentiability and where bids are flat. We follow the construction of G^R in the proof of Lemma 17 with two adjustments: (i) we substitute the left derivative G^i_{b-} for derivative G^i_b, and (ii) the differential part of the construction is separately conducted for maximal intervals (q_ℓ, q_r) on which b is strictly decreasing and continuous (as opposed to merely strictly decreasing). In this way, we are able to construct G^R for all relevant quantity and price pairs, subject to verifying monotonicity as in the above proof of Lemma 17.

Monotonicity continues to hold because G^K is monotone and whenever b is strictly decreasing and continuous, we have

$$0=-\left(v\left(q\right)-b\left(q\right)\right)\frac{G_{q}^{R}\left(q\right)}{b_{q+}\left(q\right)}-\Big(1-G^{R}\left(q\right)\Big)\leq-\left(v\left(q\right)-b\left(q\right)\right)G_{b-}^{K}\left(q;b,Q\right)-\Big(1-G^{K}\left(q;b,Q\right)\Big)\,. \tag{20}$$

For any maximal interval (q_ℓ, q_r) on which b is continuous and strictly decreasing we prove monotonicity by contradiction, as before. If there is $q \in (q_\ell, q_r)$ such that $G^R(q) > G^K(q)$, there is $\hat{q} \in [q_\ell, q_r]$ such that $G^R(\hat{q}) = G^K(\hat{q})$: even though G^K is potentially discontinuous, G^R is guaranteed to be continuous on the maximal interval in question (it is the solution to a differential equation) and G^K is monotone. At this $\hat{q}$, plugging equations 17 into inequality 20 gives

$$G_{b-}^{K}\left(\hat{q}\right)\leq\frac{G_{q}^{R}\left(\hat{q}\right)}{b_{q+}\left(\hat{q}\right)}.$$

Since b is decreasing in q, this gives

$$\begin{aligned}G_{q}^{R}\left(\hat{q}\right)&\leq G_{b-}^{K}\left(\hat{q}\right)b_{q+}\left(\hat{q}\right)\\&=G_{q+}^{K}\left(\hat{q}\right)-\mathbb{E}_{\xi}\left[K_{q+}\left(q+Q\left(b\left(q\right);\xi\right)\right)\right]\leq G_{q+}^{K}\left(\hat{q}\right).\end{aligned}$$

The final inequality follows from the fact that the exogenous supply-reserve distribution K satisfies $K_{q+} \geq 0$. Then $dG^R(q;b,Q)/dq \leq dG^K(q;b,Q)/dq+$ at $q = \hat{q}$, contradicting $G^R(q) > G^K(q)$ for some $q > \hat{q}$. The remainder of the proof follows the same steps as the original proof of Lemma 17.

[71] The left derivative of a function h at x is defined as $h_{x-}(x) = \lim_{\varepsilon\searrow 0}(h(x) - h(x-\varepsilon))/\varepsilon$. Similarly the right derivative equals $h_{x+}(x) = \lim_{\varepsilon\searrow 0}(h(x+\varepsilon) - h(x))/\varepsilon$.

H.2.3 Approximation by Strictly-Decreasing Bid Functions

We now show that we can arbitrarily approximate the first-order optimal bid b^R associated with random reserve F^R with a *strictly* decreasing bid function $\tilde{b}^R$, associated with some random reserve distribution $\tilde{F}^R$, and that the distribution of realized quantity under this approximation approximates the distribution of quantity under b^R. Then since $b^R \geq b$ and $\tilde{b}^R \approx b^R$, it follows that either $\tilde{b}^R$ approximates the revenue generated by b under reserve distribution F^R arbitrarily closely, or yields higher revenue.

Lemma 18. *Given a reserve distribution F^R with first-order optimal bid b^R and any $\varepsilon > 0$, there is a reserve distribution $\tilde{F}^R$ with a strictly decreasing first-order optimal bid $\tilde{b}^R$such that $\pi(\tilde{b}^R; \tilde{G}^R) > \pi(b^R, G^R) - \varepsilon$.*

Proof. If b^R is strictly decreasing the claim is trivially satisfied. Therefore, assume that b^R is constant on the (maximal) interval (q_ℓ, q_r). Let $\tilde{b}^R \leq b^R$ be strictly decreasing on (q_ℓ, q_r) and such that $\tilde{b}^R|_{q\notin(q_\ell,q_r]} = b^R|_{q\notin(q_\ell,q_r]}$ and $\tilde{b}^R(q_r) = \lim_{q'\searrow q_r} b^R(q')$. Let $\tilde{F}^R|_{p\geq b^R(q_\ell)} = F^R|_{p\geq b^R(q_\ell)}$. Then $\tilde{b}^R$ is first-order optimal for all $p \geq b^R(q_\ell)$ because the definition of first-order optimality is pointwise.

We now show that $\tilde{b}^R$ can be specified on $(q_\ell, q_r]$ so that (i) the probability that $q \in (q_\ell, q_r]$ is lower under $\tilde{b}^R$ than under b^R (thus the probability that $q > q_r$ is higher under $\tilde{b}^R$ than under b^R), (ii) $\tilde{b}^R$ is relatively close to b^R, and (iii) the conditional revenue under $\tilde{b}^R$, given $q \in (q_\ell, q_r]$, is not significantly below the conditional revenue under b^R. First, for a distribution F let $\Delta F \equiv F(\tilde{b}^R(q_\ell)) - F(\tilde{b}^R(q_r))$; since $\tilde{b}^R$ is first-order optimal and is strictly decreasing on $[q_\ell, q_r]$,

$$\begin{aligned}
\Delta\tilde{F}^R &= \left[\exp\left(\int_{\tilde{b}^R(q_r)}^{\tilde{b}^R(q_\ell)} \frac{1}{v\left(\tilde{\varphi}^R(y)\right) - y} dy\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R(q_r)\right) \\
&< \left[\exp\left(\ln\left[v(q_r) - \tilde{b}^R(q_r)\right] - \ln\left[v(q_r) - \tilde{b}^R(q_\ell)\right]\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R(q_r)\right) \\
&= \left(\frac{\tilde{b}^R(q_\ell) - \tilde{b}^R(q_r)}{v(q_r) - \tilde{b}^R(q_\ell)}\right) \tilde{F}^R\left(\tilde{b}^R(q_r)\right) = \left(\frac{\tilde{F}^R\left(\tilde{b}^R(q_r)\right)}{F^R\left(\tilde{b}^R(q_r)\right)}\right) \Delta F^R. \qquad (21)
\end{aligned}$$

The first inequality follows from the fact that v and $\tilde{\varphi}^R$ are strictly decreasing, and the final equality follows from the fact that b^R is first-order optimal with respect to F^R and is flat on $[q_\ell, q_r]$. Now suppose that $\tilde{F}^R(\tilde{b}^R(q_r)) < F^R(\tilde{b}^R(q_r))$; by inequality (21) it must be that $\Delta\tilde{F}^R < \Delta F^R$, and since $\tilde{F}^R(\tilde{b}^R(q_\ell)) = F^R(\tilde{b}^R(q_\ell))$ it follows that $\tilde{F}^R(\tilde{b}^R(q_r)) > F^R(\tilde{b}^R(q_r))$, a contradiction. Then $\tilde{F}^R(\tilde{b}^R(q_r)) \geq F^R(\tilde{b}^R(q_r))$, implying directly that $\Delta\tilde{F}^R \leq \Delta F^R$. Thus point (i) holds for any $\tilde{b}^R$.

Points (ii) and (iii) are shown by construction. For $\delta > 0$ sufficiently small, let $\tilde{b}^R(q_r - \delta) >$

$\tilde{b}^R(q_\ell) - \delta$. Since $\tilde{F}^R|_{p>\tilde{b}^R(q_\ell)} = F^R|_{p>\tilde{b}^R(q_\ell)}$, the expected revenue generated by bid $\tilde{b}^R$ under distribution $\tilde{F}^R$, conditional on $p > \tilde{b}^R(q_\ell)$, is identical to the expected revenue generates by bid b^R under distribution F^R, conditional on $p > \tilde{b}^R(q_\ell)$. Letting $\tilde{b}^R|_{p<\tilde{b}^R(q_r)} = b^R|_{p<\tilde{b}^R(q_r)}$, we have $||\tilde{b}^R - b^R|| < (q_r - q_\ell)\delta + (\tilde{b}^R(q_\ell) - \tilde{b}^R(q_r))\delta$ by construction. By point (i) and the analysis in the proof of Lemma 17, $\tilde{F}^R|_{p<\tilde{b}^R(q_r)} \preceq_{\text{FOSD}} F^R|_{p<\tilde{b}^R(q_r)}$, and so the expected revenue generated by bid $\tilde{b}^R$ under distribution $\tilde{F}^R$, conditional on $p < \tilde{b}^R(q_r)$, is $O(\delta)$ lower than the expected revenue generated by bid b^R under distribution F^R, conditional on $p < \tilde{b}^R(q_r)$. Finally, the utility lost when $p \in [\tilde{b}^R(q_r), \tilde{b}^R(q_\ell)]$ may be bounded in the following way. When $p \in [\tilde{b}^R(q_r), \tilde{b}^R(q_r) - \delta]$ at most quantity δ is lost (versus bid b^R), with marginal utility at most $\overline{v}$; this loss is incurred with at most probability 1, so this loss is bounded above by $\overline{v}\delta$. When $p \in [\tilde{b}^R(q_\ell) - \delta, \tilde{b}^R(q_\ell)]$, the quantity lost (versus bid b^R) is at most $(q_r - q_\ell) < \overline{Q}$, with marginal utility at most $\overline{v}$. However, the probability that this quantity is lost is bounded by

$$
\begin{aligned}
&\tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)\right) - \tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right) - \delta\right) \\
&= \left[\exp\left(\int_{\tilde{b}^R(q_\ell)-\delta}^{\tilde{b}^R(q_\ell)} \frac{1}{v\left(\tilde{\varphi}^R\left(y\right)\right) - y} dy\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right) - \delta\right) \\
&\leq \left[\exp\left(\int_{\tilde{b}^R(q_\ell)-\delta}^{\tilde{b}^R(q_\ell)} \frac{1}{v\left(q_r\right) - y} dy\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)\right) \\
&= \left[\exp\left(\ln\left[v\left(q_r\right) - \left(\tilde{b}^R\left(q_\ell\right) - \delta\right)\right] - \ln\left[v\left(q_r\right) - \tilde{b}^R\left(q_\ell\right)\right]\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)\right) \\
&= \left(\frac{\delta}{v\left(q_r\right) - \tilde{b}^R\left(q_\ell\right)}\right) \tilde{F}^R\left(\tilde{b}^R\left(q_\ell\right)\right).
\end{aligned}
$$

This probability is thus bounded above by a term linear in δ; indeed $v(\cdot) > b(\cdot)$ for all units which are received with strictly positive probability (Lemma 8) and hence $v(q_r) - \tilde{b}^R(q_\ell) = v(q_r) - b^R(q_r) > 0$. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that the revenue generated by the first-order optimal bid function $\tilde{b}^R$ under reserve distribution $\tilde{F}^R$ is no more than λ below the revenue generated by the first-order optimal bid function b^R under reserve distribution F^R. □

The above two lemmas imply the following approximation result:

Lemma 19. *Given any best response bid curve $b(\cdot)$ and any $\varepsilon > 0$, there is a massless reserve distribution $\tilde{F}^R$ with strictly decreasing first-order best response $\tilde{b}^R$ such that such that the first order best response to F^R generates no more than λ less revenue than $b(\cdot)$.*

H.2.4 An Auxiliary Uniform-Price Auction with Known Values

We maintain the auxiliary assumption that the bidder whose response we analyze has no private information. Having shown that we can restrict attention to random reserve, we continue the analysis by showing that any strictly decreasing first-order optimal bid $\tilde{b}^R$generates strictly less revenue than some uniform-price auction (Theorem 20), which we then bound by pay-as-bid revenue in the next and final subsection, where we also drop the no-private-information assumption.

Lemma 20. *Given a massless distribution of reserve prices F^R and a strictly decreasing first-order optimal bid b^R, there is a distribution of reserve prices $\hat{F}^R$ such that the uniform-price auction under reserve distribution $\hat{F}^R$ generates the same expected revenue as the pay-as-bid auction with first-order optimal bid b^R and reserve distribution F^R.*

Proof. We may assume that the support of the distribution F^R is contained in the support of marginal values on units the bidder can win. Indeed, our assumptions on the utility imply that this support is convex and thus reserves outside of support are either above or below it. Probability mass of reserve prices above the support can be arbitrarily shifted to reserves in the support, increasing expected revenue, and similarly for probability mass of reserve prices below the support of marginal values; the latter operation might create an atom at the bottom of the support, but as we have seen in the proofs for Section 4 (cf. Appendix E.4), this atom does not affect the bidder's best response behavior. Under these assumptions, truthful reporting, $b \equiv v$, is the essentially unique equilibrium in a uniform-price auction with random reserve drawn from F^R. Under a random reserve distribution, each bidder's problem is a single-person decision problem. Because demand at a particular price does not affect outcomes at other prices, at each price bidders should demand a utility-maximizing quantity. As b is strictly decreasing and first-order optimal, φ and φ_p are well-defined and $v(\hat{\varphi}^R(p)) = p$ at all relevant prices p.

Revenue in the pay-as-bid auction under reserve distribution F^R is

$$\mathbb{E}\left[\pi\right] = \int_{\underline{b}}^{\bar{b}} \left(p\varphi^R\left(p\right) + \int_p^{\bar{b}} \varphi^R\left(x\right) dx \right) f^R\left(p\right) dp.$$

Define $\hat{F}^R$ so that

$$\hat{F}^R\left(v\left(\varphi^R\left(p\right)\right)\right) = F^R\left(p; s\right).$$

By construction, $\hat{F}^R_p(v(\varphi^R(p)))v_q(\varphi^R(p))\varphi^R_p(p) = F^R_p(p)$. Additionally, $\operatorname{Supp} \hat{F}^R = [\underline{p}, \overline{v}]$, and in a uniform-price auction with reserve distribution $\hat{F}^R$, it is weakly optimal for the bidder to submit truthful bids for all quantities q such that $v(q) \in [\underline{b}, \overline{v}]$. The revenue in this auction is

$$\mathbb{E}\left[\hat{\pi}\right] = \int_{\underline{b}}^{\overline{v}} pv^{-1}\left(p\right)\hat{F}_p^R\left(p\right)dp.$$

Apply a change of variables, so that $p = \hat{v}(\varphi^R(p'))$. Then $dp = v_q(\varphi^R(p'))\varphi_p^R(p')dp'$. Since $\varphi^R(\overline{p}) = 0$, this gives

$$\begin{aligned}
\mathbb{E}\left[\hat{\pi}\right] &= \int_{\underline{b}}^{\bar{b}} v\left(\varphi^R\left(p'\right)\right)v^{-1}\left(v\left(\varphi^R\left(p'\right)\right)\right)\hat{F}_p^R\left(v\left(\varphi^R\left(p'\right)\right)\right)v_q\left(\varphi^R\left(p'\right)\right)\varphi_p^R\left(p'\right)dp' \\
&= \int_{\underline{b}}^{\bar{b}} v\left(\varphi^R\left(p'\right)\right)\varphi^R\left(p'\right)F_p^R\left(p'\right)dp'.
\end{aligned}$$

Then compare,

$$\begin{aligned}
\mathbb{E}\left[\pi\right] - \mathbb{E}\left[\hat{\pi}\right] &= \int_{\underline{b}}^{\bar{b}}\left(p\varphi^R\left(p\right) + \int_p^{\bar{b}}\varphi^R\left(x\right)dx\right)F_p^R\left(p\right) - v\left(\varphi^R\left(p\right)\right)\varphi^R\left(p\right)F^R\left(p\right)dp \\
&= \int_{\underline{b}}^{\bar{b}}\left(-\left(v\left(\varphi^R\left(p\right)\right) - p\right)\varphi^R\left(p\right) + \int_p^{\bar{b}}\varphi^R\left(x\right)dx\right)F_p^R\left(p\right)dp \\
&= \int_{\underline{b}}^{\bar{b}}\left(-\left[\frac{F^R\left(p\right)}{F_p^R\left(p\right)}\right]\varphi^R\left(p\right) + \int_p^{\bar{b}}\varphi^R\left(x\right)dx\right)F_p^R\left(p\right)dp \\
&= -\int_{\underline{b}}^{\bar{b}}\varphi^R\left(p\right)F^R\left(p\right)dp + \int_{\underline{b}}^{\bar{b}}\int_p^{\bar{b}}\varphi^R\left(x\right)dxF_p^R\left(p\right)dp \\
&= -\int_{\underline{b}}^{\bar{b}}\varphi^R\left(p\right)F^R\left(p\right)dp + \left[\int_p^{\bar{b}}\varphi^R\left(x\right)dxF^R\left(p\right)\right]\Bigg|_{p=\underline{b}}^{\bar{b}} + \int_{\underline{b}}^{\bar{b}} q^R\left(p\right)F^R\left(p\right)dp = 0.
\end{aligned}$$

The transition from the second line to the third comes from the bidder's first-order condition under random reserve. Then the uniform-price auction with reserve distribution $\hat{F}^R$ generates the same revenue as the pay-as-bid auction with reserve distribution F^R and first-order optimal bid b^R. □

H.2.5 Revenue Dominance of Deterministic Mechanisms with Known Values

Our previous lemmas imply that, when a bidder has no private information, the seller can weakly improve the revenue obtained from this bidder by implementing a uniform price auction with a random reserve price. These results are independent of opponent strategies in the pay-as-bid auction. Furthermore, we argued above that when the bidder participates in an auction with a random reserve price (and sufficiently large fixed supply) her best response is independent of her opponents' strategies. Thus, if the seller knew each bidder's private information, they could improve revenue by implementing a bidder-specific uniform-price auction with a random reserve price.

We are now ready to conclude the proof of Lemma 2 by showing that the above uniform price auction generates less revenue than a deterministic pay-as-bid auction, still in the auxiliary environment in which bidders have no asymmetric information (equivalently, when their information is known to the seller).

Proof. Focusing on one bidder and putting together Lemmas 17, 18, and 20 we can conclude that for any $\lambda > 0$ and any random elastic supply in a pay-as-bid auction, there is a uniform-price auction with random reserve that raises from the bidder we focus on at least the pay-as-bid auction revenue minus λ. As we have seen in the first paragraph of the proof of Lemma 20, in this uniform-price auction we may assume that the bidder bids their true marginal value curve (at all prices in the support of the random reserve distribution), and ex post revenue is always weakly below monopoly revenue. It follows that the uniform-price auction's revenue is maximized by selling the deterministic monopoly quantity with an appropriate reserve price. By Theorem 5, this revenue is equivalent to what the seller would obtain by implementing a pay-as-bid auction for the (deterministic) monopoly quantity, with or without a reserve price. Thus, to maximize the revenue obtained from a single bidder whose information is known to the seller, it is optimal to deterministically sell the bidder the monopoly quantity.

Because bidders are symmetric, it follows that it is optimal to deterministically sell them the aggregate monopoly quantity (note that the equilibrium price will be the monopoly price as long as the seller sets the reserves weakly below it). $\square$

H.3 Proof of Theorem 9 (Optimality of Deterministic Mechanisms)

Proof. If the seller knows the bidders' common signal s, the optimal quantity in a pay-as-bid auction is $Q^\star(s) \in \arg\max_{Q \leq Q^{\max}} Qv(Q/n; s)$, and in the unique equilibrium of this pay-as-bid auction, $p^\star(Q^\star(s); s) = v(Q^\star(s)/n; s)$.[72] Let $\mathcal{Q} : \mathbb{R}_+ \to \mathbb{R}_+$ be a supply curve, where $\mathcal{Q}(p) = \inf\{Q^\star(s) \colon p^\star(s) > p\}$. Bidder values are regular, so $\mathcal{Q}$ is increasing. Then equilibrium in the pay-as-bid auction with supply curve $\mathcal{Q}$ is such that for any bidder signal s, $p(Q^\star(s); s) = v(Q^\star(s)/n; s)$, and revenue is maximized for each type independently. $\square$

H.4 Proof of Theorem 10 (Revenue Dominance of Pay-as-Bid)

Proof. Consider the (deterministic) optimal supply curve derived in Theorem 9. Given this supply curve, there is an equilibrium of the uniform-price auction in which bidders submit truthful bids; furthermore, because supply is deterministic the market-clearing price must

[72] Equilibrium uniqueness is established in Theorem 8.

be weakly below each bidder's marginal value for their marginal unit, hence truthful bids provide an upper bound on uniform-price revenue. As in the pay-as-bid auction, regularity allows us to compare auction revenues for an observable realization of the bidder-common signal s. The market clearing price and quantity correspond then to the monopoly solution, and maximal revenue in this equilibrium of the uniform-price auction is equivalent to revenue in the unique equilibrium of the optimal pay-as-bid auction. No higher revenue is feasible in the uniform-price auction—even with different distribution over supply-reserve—because for known s the revenue is bounded above by monopoly revenue. $\square$