Ordinal Simplicity and Auditability in Discrete Mechanism Design*

Marek Pycia† M. Utku Ünver‡

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Abstract

Designing mechanisms for environments without transfers, market designers usually restrict attention to ordinal mechanisms. Ordinal mechanisms are simpler for both designers and participants but miss potentially welfare-relevant information. Under what conditions focusing on ordinal mechanisms is without loss? We show that, in general, all group strategy-proof mechanisms are ordinal. All mechanisms maximizing an Arrovian social welfare function are ordinal; in a large class of environments, such Arrovian efficiency is implied by Pareto efficiency and a simple auditability condition. Strategy-proof mechanisms that are simple to audit are also ordinal. As applications, we characterize important classes of mechanisms in public choice as well as single-unit-demand and multiple-unit-demand allocation of private goods.

Keywords: Ordinality, simplicity, strategy-proofness, auditability, non-bossiness, Pareto efficiency, welfare maximization, Arrovian efficiency, rich domains, public choice, house allocation, single-unit demand, multi-unit demand.

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†University of Zurich, Department of Economics, e-mail: marek.pycia@econ.uzh.ch.

‡Boston College, Department of Economics, e-mail: unver@bc.edu.

1 Introduction

Microeconomic theory has informed the design of many markets and other institutions. In environments in which transfers are not used or are prohibited, the implemented or proposed mechanisms usually elicit only ordinal preference information from participants. Examples include public choice mechanisms such as majority or plurality voting (Moulin, 1980), medical resident-hospital matching mechanisms (Roth and Peranson, 1999), school choice and college admission mechanisms based on deferred acceptance, top trading cycles, or serial dictatorship (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003), the mechanisms allocating dormitory rooms in colleges (Abdulkadiroğlu and Sönmez, 1999), and many mechanisms for allocation and exchange of transplant organs, such as kidneys and livers (Roth, Sönmez, and Ünver, 2004; Ergin, Sönmez, and Ünver, 2020). Correspondingly, the economic literature on mechanisms without transfers generally focused on ordinal mechanisms.

A mechanism is ordinal if for all utility profiles that represent the same ordinal preferences, participants obtain utility equivalent outcomes. Taking ordinality for granted comes at the expense of welfare relevant cardinal information.[1] The reliance on only ordinal information is an important property of a mechanism. As observed as early as in Bogomolnaia and Moulin (2001), ordinality simplifies the mechanism as the participants need only to submit ordinal rankings over sure outcomes.[2] Ordinality also implies that the range of the mechanism is discrete.

When the focus on ordinal mechanisms is without loss of generality? We show that this is so when the designer imposes natural incentive properties or strong efficiency properties. Incentive compatibility and efficiency are the two central desiderata in the development of allocation mechanisms.[3]

Our first insight is that the ordinality restriction is without loss in problems in which the desired mechanisms are group strategy-proof (Theorem 1). Group strategy-proofness requires the mechanism to be immune to group deviations in which there is an agent who is strictly better off while other agents in the group are weakly better off. Immunity to group deviations is important in many environments without transfers in which participants can coordinate (cf. Pathak and Sönmez (2008) and Barberà, Berga, and Moreno (2016)). Theorem 1 shows slightly more: every individually strategy-proof and non-bossy mechanism is ordinal. A mechanism is individually strategy-proof (or strategy-proof, for brevity) if, for any reports by other individuals, reporting her true utility report leads to the mechanism outcome being weakly better for an individual than any other report. A mechanism is non-bossy if, whenever an agent cannot change her utility from

[1]That point was made, e.g., by Miralles (2009), Abdulkadiroğlu, Che, and Yasuda (2011), and Featherstone and Niederle (2016). Mechanisms that elicit cardinal information include pseudomarket (token-money) mechanisms of Hylland and Zeckhauser (1979), Budish (2009), and He et al. (2018), as well as the linear programming mechanism of Nguyen, Peivandi, and Vohra (2016).

[2]Eliciting cardinal information in settings without money is a challenge as has been observed in the studies of pseudomarkets, cf. Budish and Kessler (2022).

[3]For instance, in private good allocation environments, Bogolomania and Moulin (2004) write that "the central question of that literature is to characterize the set of efficient and incentive compatible (strategy-proof) assignment mechanisms;" a view echoed by Barberà, Berga, and Moreno (2016) and others.

the outcome chosen by changing her utility report, then she cannot change any other individual's either. We establish these ordinality insights for general private-value environments and are agnostic as to whether monetary transfers are allowed. The main motivation for our study comes however from market design without transfers because conditions such as group strategy-proofness are satisfied by many mechanisms in environments without transfers, but are rather restrictive in environments with transfers.

Our second insight is that ordinality is implied by Arrovian social welfare maximization. A mechanism is Arrovian efficient if it always chooses the top alternative of an Arrovian social welfare function.[4] Maximizing welfare is a standard way to strengthen Pareto efficiency in many economic problems. Pareto efficiency is a weak efficiency concept; while interpersonal utility comparisons are not needed for Pareto efficiency, it only gives a lower bound for what can be achieved through desirable mechanisms. In consequence, welfare economics—starting with Bergson (1938), Samuelson (1947), and Arrow (1963)—introduced stronger efficiency concepts requiring an efficient outcome to be the maximum of a social ranking of outcomes; an idea later named as resoluteness.[5] For instance, Arrow (1963) discussed the partial ordering of outcomes given by Pareto dominance, and observed:

> But though the study of maximal alternatives is possibly a useful preliminary to the analysis of particular social welfare functions, it is hard to see how any policy recommendations can be based merely on a knowledge of maximal alternatives. There is no way of deciding which maximal alternative to decide on.

Despite Arrow's critique, the literature on no-transfers allocation mechanisms focused on Pareto efficiency, and one of the contributions of our paper is to bring Arrow's program of resolute efficiency to the study of these allocation mechanisms.

We also contribute to the Arrow's program by reinterpreting his independence-of-irrelevant-alternatives postulate posited for preference aggregation as simple auditability in mechanism design: in order to falsify a proposed mechanism outcome, it is sufficient to verify just one individual's ranking of just two alternatives: the outcome and challenger alternative.[6] This auditability

[4]A social welfare function maps utility profiles to social rankings. A social welfare function is Arrovian if, it satisfies the postulates of Arrow (1963): resoluteness, (strong) Pareto, and independence of irrelevant alternatives. A social welfare function is resolute if it has a unique social maximum for every profile of utilities. It satisfies the (strong) Pareto postulate if, whenever two Pareto-comparable alternatives are ranked socially, the Pareto-dominant one is ranked above the Pareto-dominated one. A social welfare function satisfies independence of irrelevant alternatives if, given any two utility profiles and any two alternatives that are socially comparable under both profiles such that all individuals rank the two alternatives in the same way under both utility profiles, then the social ranking of the two alternatives is the same under both profiles.

[5]Resoluteness has been a standard property in social choice since its conception and its failure is at the core of the Condorcet paradox, see e.g. Black (1948) and Campbell and Kelly (2003). See Austen-Smith and Banks (1999) for the role of resoluteness in political science, and Zwicker (2016) for a recent survey of canonical social choice results such as Gibbard (1973)-Satterthwaite (1975) Theorem that implicitly or explicitly involve resoluteness.

[6]Our simple auditability is a weak version of the independence-of-irrelevant-alternatives postulate. For other weak versions of the independence postulate see, e.g., Fleurbaey and Maniquet (2008a) and Fleurbaey and Maniquet (2008b) who study the independence postulate in the context of fairness. For other concepts of auditability see, e.g., Akbarpour and Li (2020), Woodward (2020), Hakimov and Raghavan (2020), Möller (2022), and Grigoryan and Möller (2023).

property is independently attractive as it allows to falsify the mechanism outcome with a de minimis amount of information, thus economizing on verification costs and preserving some of the privacy of participants' private information. The mechanism can be falsified (audited) with this very limited information provided the auditor (or court) knows where to look. Knowing where to look is not necessarily to be taken for granted but it is much easier than the verification in two important contexts: when (i) the auditor receives information from a whistleblower and (ii) when observing (or informally analyzing) the problem is easier than formally proving (verifying) the violation in front of a court. The difference between observability and verifiability is ubiquitous in practice and gave raise to the large literature on incomplete contracts, for a discussion see, e.g., Hermalin and Katz (1991) and Maskin and Tirole (1999).[7] While Arrow introduced independence of irrelevant alternatives as a normative postulate for preference aggregation, our re-interpretation in terms of auditability for mechanism design is more positive. As it is only imposed on the comparisons between the mechanism's outcome and the challenger alternative, auditability is formally weaker than independence of irrelevant alternatives; still every auditable and Pareto-efficient mechanism is Arrovian efficient.

Our Theorem 2 shows that every Arrovian efficient—and hence every auditable and Pareto efficient—mechanism is ordinal. Furthermore, Proposition 3 shows that every group strategy-proof mechanism is auditable and every auditable mechanism is non-bossy; with the reverse implications failing as we illustrate via counterexamples. Taking into account Theorem 1, we obtain that every strategy-proof and auditable mechanism is also ordinal (Theorem 3).

We develop the connection between group incentives and ordinality for all economic environments with private values, including discrete environments both with and without transfers. For farther general results we focus on a still very broad class of discrete environments, merely imposing a natural richness assumption on preference domains. Richness is a substantial weakening of the Arrovian universal strict domain assumption and it is satisfied in many practically and theoretically relevant economic domains including matching with unit or multi-unit demand under strict preferences, and allocation of discrete resources without compensating transfers.[8] In particular, we show that, in any rich (cardinal) domain, group strategy-proofness not only implies but is equivalent to the conjunction of strategy-proofness and non-bossiness (Proposition 1), substantially generalizing such an equivalence established by Pápai (2000) for ordinal house allocation.[9]

[7] For the role of whistleblowers see, e.g., Naritomi (2019). For advantages of using limited information see, e.g., Mount and Reiter (1996), Segal (1999), Hurwicz and Reiter (2006), and Nisan and Segal (2006). For the literature on privacy in mechanism design see the recent survey Pai and Roth (2018).

[8] We adapt the definition of richness from Pycia and Troyan (2019); for an earlier use of an equivalent assumption see Takamiya (2007).

[9] We use the same concept of group strategy-proofness as earlier studies of no-transfer allocation such as Pápai (2000), Ehlers (2002), and Pycia and Ünver (2017). A weaker concept of group strategy-proofness—which only requires immunity against group deviations in which each agent in the group is strictly better off—has also been studied. E.g., Dubins and Freedman (1981) recognized that the deferred acceptance mechanism is weakly group-strategy-proof for the proposing side; this mechanism is not group strategy-proof. Barberà, Berga, and Moreno (2010) and Barberà, Berga, and Moreno (2016) discuss the difference between the two group strategy-proofness concepts and the canonical mechanisms that satisfy them. They also show that, in many preference domains, if a mechanism satisfies certain

Our main results allow us to derive many new natural characterizations of cardinal mechanisms in important economic environments. For public choice with the universal strict preference domain, we show that strategy-proofness implies auditability (but not vice versa, Proposition 5), implying that Arrovian efficiency and Pareto efficiency are equivalent conditions for strategy-proof mechanisms (Corollary 1).[10]

For the allocation of objects to individuals with unit demand and strict preferences—often referred to as house allocation problems—we combine our results and the results of Pycia and Ünver (2017) to fully characterize the class of group strategy-proof and Pareto efficient cardinal mechanisms as equal to the class of strategy-proof, Pareto efficient, and auditable cardinal mechanisms, and as further equal to Pycia and Ünver's (2017) class of *trading cycles mechanisms*.[11] We also consider cardinal strategy-proof mechanisms that are efficient with respect to complete Arrovian social welfare functions, i.e., those that rank all alternatives; in house allocation an alternative us a matching.[12] We show that the class of cardinal mechanisms that are strategy-proof and Arrovian efficient with respect to a complete social welfare function consists of mechanisms that we call *almost sequential dictatorships* (Theorems 5 and 6). An almost sequential dictatorship combines the ideas of *sequential dictatorship* and *majority voting* between only two possible outcomes. Dictatorships are the benchmark strategy-proof and efficient mechanisms in many areas of economics. When there are three or more alternatives, Gibbard (1973) and Satterthwaite (1975) have shown that all strategy-proof and unanimous public choice mechanisms are dictatorial.[13] With two alternatives there are other mechanisms that are strategy-proof and unanimous; majority voting being the primary example. The class of almost sequential dictatorships combines both of these special mechanisms. Despite these parallels, we find it surprising that a version of Gibbard (1973) and Satterthwaite (1975) theorem is true in our environment because—in stark contrast to the environments where this question was previously studied—ours allows many strategy-proof (and even group strategy-proof) and Pareto-efficient mechanisms that are not dictatorial.[14]

Sequential dictatorships also play focal role in the allocation of objects to individuals with multi-unit demand. We combine our results with those of Hatfield (2009) to show that all strategy-

monotonicity and respectfulness conditions, then this mechanism is individually strategy-proof if, and only if, it is weakly group strategy-proof.

[10]Our results also imply further rich-domain equivalences between canonical axioms of mechanism design, cf. Proposition 4 and Appendices A and C.

[11]Another characterization of a more demanding auditability-like concept (credibility) was provided by Akbarpour and Li (2020), who studied single-object allocation with transfers and showed that strategy-proofness, efficiency, and credibility imply that the mechanism is an ascending clock auction.

[12]Knowing the complete social welfare function allows one to determine the socially optimally alternative in any subset of alternatives, hence facilitating analysis of welfare under different resource constraints.

[13]Dasgupta, Hammond, and Maskin (1979) extended this result to more general social choice models. Satterthwaite and Sonnenschein (1981) extended it to public goods economies with production. Zhou (1991) extended it to pure public goods economies. In exchange economies, Barberà and Jackson (1995) showed that strategy-proof mechanisms are Pareto inefficient.

[14]Dictatorships are not in general Pareto efficient and thus not Arrovian-efficient in the house allocation domain. Sequential dictatorships modify the dictatorship idea to reestablish Pareto (and Arrovian) efficiency, for their study see, e.g. were studied by, e.g., Satterthwaite and Sonnenschein (1981). In house allocation, the above discussed Trading Cycle mechanisms—a class much more flexible than sequential dictatorships—are strategy-proof and Pareto-efficient.

proof, Pareto efficient, and auditable cardinal mechanisms (or equivalently, all group strategy-proof and Pareto efficient mechanisms, and all strategy-proof and Arrovian efficient mechanisms) are sequential dictatorships.

While it may not be so ex post, given the literature it is ex ante surprising that simple incentive or efficiency conditions that we study imply that the mechanisms are ordinal. The first related result is Gibbard (1977) who showed that mechanisms that are strategy-proof and Pareto efficient on a universal domain of utilities are serial dictatorships and hence ordinal; in contrast our ordinality results do not hinge on strong domain assumptions. Satterthwaite and Sonnenschein (1981) showed that, in public good economies, strategy-proofness, non-bossiness, and additional assumptions on the mechanism and the utility space imply that a cardinal mechanism is a serial dictatorship, and hence ordinal; by focusing directly on ordinality we are able to relax their additional assumptions. More recently, Carroll (2018) showed that if a social choice correspondence mapping ordinal preferences to outcomes is implementable in ex-post equilibrium in all interdependent-value extensions of the ordinal preferences, then the correspondence is implementable by an ordinal mechanism; in contrast we show that incentive-compatibility assumptions imply ordinality in private value settings, without imposing any assumptions on interdependent-value extensions. Ehlers, Majumdar, Mishra, and Sen (2020) showed that strategy-proof mechanisms satisfying certain continuity conditions are ordinal; our analysis does not not rely on any continuity conditions.[15] Calsamiglia, Martinez-Mora, and Miralles (2021) proved that affine-invariant mechanisms whose outcomes do not depend on the presence and valuation of outside options are ordinal; our analysis does not rely on affine invariance or the robustness to the presence of outside options.

Our paper connects the literature on discrete mechanism design and the literature on Arrovian preference aggregation. In addition to the papers mentioned above, Muller and Satterthwaite (1977) interpreted the monotonicity property of mechanisms as a stronger version of Arrow's independence of irrelevant alternatives postulate and showed that in the universal strict preference domain, an ordinal mechanism is monotonic if, and only if, it is strategy-proof. Takamiya (2007) extended this result to ordinal rich domains. Our interpretation of independence of irrelevant alternatives postulate for mechanisms, auditability, is strictly weaker than monotonicity even in the universal strict domain. An extensive literature extends Arrow's program to economic domains and focuses on determining the class of preference domains in which Arrow's result holds, i.e., economic domains in which all complete Arrovian social welfare functions are dictatorial (see e.g. Kalai, Muller, and Satterthwaite (1979) and Le Breton and Weymark (2011)) as well as those in which Arrow's postulates lead to non-dictatorial social welfare functions (Gaertner (2001) provides a survey).[16] For instance, Sethuraman, Piaw, and Vohra (2003) develop a novel approach to the Arrovian program by reformulating its postulates as linear constraints on social welfare functions and illustrate its usefulness by applying it to many questions in axiomatic mechanism

[15]They recognize that in the pure public choice environment, like the one studied by Gibbard (1977), their continuity conditions can be relaxed.

[16]See also Bordes and Le Breton (1989, 1990b,a); Bordes, Campbell, and Le Breton (1995).

design, including providing a novel proof of Muller and Satterthwaite's theorem. Our results on ordinal simplicity of cardinal mechanisms are complementary to this literature. We also go beyond this literature by moving beyond the dictatorship question.[17]

Our paper also contributes to the literature on characterizations of dominant strategy mechanisms in important economic domains, particularly multi-unit and single-unit demand allocation problem. For multi-unit demand, Hatfield (2009) showed that all strategy-proof, nonbossy, and Pareto efficient ordinal mechanisms are sequential dictatorships. For single-unit demand, Pycia and Ünver (2017) characterized group-strategy-proof and Pareto-efficient ordinal mechanisms in the standard domain of strict preferences with unit demand and Root and Ahn (2020) characterized properties of such mechanisms allowing for constraints and providing a synthetic treatment of many social choice domains; see also Barberà (1983) and Pápai (2000) who laid the foundations for this line of research. Ehlers (2002) characterized group-strategy-proof and Pareto-efficient ordinal mechanisms in a maximal domain of weak preferences for which such mechanisms exist and proves a general impossibility result for the domain of all weak preferences.[18] Ma (1994) characterized the class of ordinal strategy-proof, individually rational, and Pareto-efficient mechanisms, and Sönmez and Ünver (2010a) obtained a characterization in mixed ownership economies with both commonly and privately owned good. Ma's characterization has been extended by Pycia and Ünver (2017) and Tang and Zhang (2015) to richer single-unit demand, by Pápai (2007) to multi-unit demand models, and by Pycia (2016) to settings with network constraints.[19]

Sequential dictatorships have not been studied extensively with unit demand for goods, although their special cases have been. In a *serial dictatorship* (also known as a *priority mechanism*), the same individual chooses next regardless of which house the current individual picks. Svensson (1994) formally introduced and studied serial dictatorships first; Abdulkadiroğlu and Sönmez (1998) studied a probabilistic version of them where the order of individuals is determined uniformly randomly; Svensson (1999) and Ergin (2000) characterized them using plausible axioms. Allowing for outside options, Pycia and Ünver (2022) characterized a subclass of sequential dictatorships different from serial dictatorships. With multiple-house demand under responsive preferences, Hatfield (2009) showed that sequential dictatorships are the only strategy-proof, nonbossy, and Pareto-efficient mechanisms, and Pápai (2001) characterized the sequential dictatorships through the properties of strategy-proofness, non-bossiness, and citizen sovereignty (see also Klaus and Miyagawa, 2002). In a general model allowing both the cases with and without

[17]We focus on the strong Pareto postulate (in line with much of contemporary economics) instead of the weak Pareto postulate of the Arrovian program literature. E.g., see Sen (1969) for an early example, and Bordes and Le Breton (1990a) for matching domains. The strong Pareto postulate calls an alternative Pareto dominated if there is one agent who strictly prefers a challenger alternative while other agents prefer it weakly; the weak Pareto postulate calls an alternative dominated only if all agents strictly prefer another alternative. The difference matters as the focal dictatorship mechanisms are weakly Pareto efficient but in many domains they fail to be Pareto efficient in the strong sense.

[18]Most of the literature on house allocation—including our paper—is not affected by Ehlers' impossibility result because it analyzes environments in which individuals' preferences are strict. Our concept of partial social ranking is different from Ehlers' allowing only certain weak preferences over assigned houses; Ehlers' work is not concerned with social rankings of outcomes and we have equivalence classes for indifferences.

[19]See Sönmez and Ünver (2010b) for a survey of additional results in this literature.

transfers, Pycia and Troyan (2019) showed that a broad class closely resembling sequential dictatorships are precisely the mechanisms that are strongly obviously strategy-proof in their sense; see also Li (2017) and Pycia (2019). For characterizations of random serial dictatorships in terms of incentives, efficiency, and fairness see Liu and Pycia (2011) and Pycia and Troyan (2019). Root and Ahn (2020) characterized the constrained social choice domains in which generalized sequential dictatorships are the only group strategy-proof and Pareto-efficient mechanisms. As an application of their general theorem, they characterize sequential dictatorships as the only mechanisms which are group strategy-proof and Pareto efficient in the roommates problem.

2 Model

2.1 Environment

Let $\mathcal{I}$ be a finite set of **individuals** and $\mathcal{A}$ be a finite set of social **alternatives**. Each individual i has a **utility function** $v_i : \mathcal{A} \to \mathbb{R}$ inducing a complete, reflexive, and transitive binary relation—her weak preference relation—denoted by $\succcurlyeq_i$ over $\mathcal{A}$. We denote its strict (i.e., anti-symmetric) part by $\succ_i$ and indifference (i.e., symmetric) part by $\sim_i$. Let $\mathbf{V}_i$ be the domain of utility functions for agent i and $\mathbf{R}_i$ be the induced domain of preference relations, and let $\mathbf{V}_{\mathcal{J}}$ and $\mathbf{R}_{\mathcal{J}}$ denote the respective Cartesian products $\bigtimes_{i\in\mathcal{J}} \mathbf{V}_i$ and $\bigtimes_{i\in\mathcal{J}} \mathbf{R}_i$ for any $\mathcal{J} \subseteq \mathcal{I}$. Any profile $v = (v_i)_{i\in\mathcal{I}}$ from $\mathbf{V} = \mathbf{V}_{\mathcal{I}}$ is called a **utility profile** and $\succcurlyeq = (\succcurlyeq_i)_{i\in\mathcal{I}}$ from $\mathbf{R} = \mathbf{R}_{\mathcal{I}}$ is called a **preference profile**. For every $v \in \mathbf{V}$ and its induced preference profile $\succcurlyeq$ and $\mathcal{J} \subseteq \mathcal{I}$, let $v_{\mathcal{J}} = (v_i)_{i\in\mathcal{J}} \in \mathbf{V}_{\mathcal{J}}$ and $\succcurlyeq_{\mathcal{J}} = (\succcurlyeq_i)_{i\in\mathcal{J}} \in \mathbf{R}_{\mathcal{J}}$ be their restrictions to $\mathcal{J}$. Throughout the paper, we fix $\mathcal{I}$ and $\mathcal{A}$, and thus, a problem is identified with its utility profile.

Many, though not all, of our results are stated for domains of utility profiles that are rich in the following sense. For every individual i there is an exogenous **equivalence relation** $\equiv_i$ on alternative set $\mathcal{A}$. We say that the domain $\mathbf{V}_i$ is **rich** if the following two conditions are satisfied:

1. If for any two alternatives a and b we have $a \equiv_i b$, then for every $v_i \in \mathbf{V}_i$ we have $v_i(a) = v_i(b)$.

2. If no alternatives in $\mathcal{A}' \subseteq \mathcal{A}$ are $\equiv_i$-equivalent, then all utility functions that lead to strict preferences on $\mathcal{A}'$ belong to $\mathbf{V}_i$ and any utility function in $\mathbf{V}_i$ leads to strict preferences on $\mathcal{A}'$.

The last condition eliminates redundancies in our description of the preferences over alternatives. For instance, in house allocation, each social alternative a is a matching between individuals and objects from some set and $a \equiv_i b$ if, and only if, the object matched to i is the same under a and b.

Effectively, $\mathbf{V}_i$ is the universal strict preference domain respecting $\equiv_i$-equivalence classes.[20] We

[20] An analogue of the richness concept for ordinal preference domains was introduced by Pycia and Troyan (2019). They allowed exogenous structural preference relations that can be but not necessarily are equivalence relations. In their terminology, our setting corresponds to no-transfer environments. The class of environments studied in Takamiya (2007) corresponds to rich ordinal domains. Substantively different richness concepts were studied, e.g., by Dasgupta, Hammond, and Maskin (1979), Pycia (2012), and Barberà, Berga, and Moreno (2016).

say that the utility profile domain $\mathbf{V}$ is **rich** if $\mathbf{V}_i$ is a rich utility domain for every $i \in \mathcal{I}$. In the rest of the paper, we assume that $\mathbf{V}$ is a rich utility profile domain for a fixed equivalence relation profile $(\equiv_i)_{i\in\mathcal{I}}$. Examples of rich domains include public good allocation and public choice in universal strict domain, object allocation (as known as house allocation) with unit or multi-unit demand and no transfers.

2.2 Mechanisms and Their Properties

A (direct) **cardinal mechanism** (or simply a **mechanism**) is a mapping $\varphi : \mathbf{V} \to \mathcal{A}$ that assigns an alternative for every utility profile (or, equivalently, for every problem). We denote the outcome of mechanism φ for a utility profile $v \in \mathbf{V}$ as $\varphi[v]$.

A mechanism is strategy-proof if for every individual, she weakly prefers the outcome when she is truthful to the outcome under any untruthful revelation of her utility function. Formally, a mechanism φ is **strategy-proof** if for every $v \in \mathbf{V}$, there exists no $i \in \mathcal{I}$ and $v'_i \in \mathbf{V}_i$ such that

$$v_i(\varphi[v'_i, v_{-i}]) > v_i(\varphi[v_i, v_{-i}]).$$

A mechanism is group strategy-proof if there is no group of individuals that can misstate their utility functions in a way such that each one in the group is weakly better off and at least one individual in the group is strictly better off, irrespective of the utility profile of the individuals not in the group. Formally, a mechanism φ is **group strategy-proof** if for every $v \in \mathbf{V}$, there exists no $\mathcal{J} \subseteq \mathcal{I}$ and $v'_{\mathcal{J}} \in V_{\mathcal{J}}$ such that

$$v_j(\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}]) \geqslant v_j(\varphi[v_{\mathcal{J}}, v_{-\mathcal{J}}]) \text{ for every } j \in \mathcal{J},$$

and

$$v_i(\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}]) > v_i(\varphi[v_{\mathcal{J}}, v_{-\mathcal{J}}]) \text{ for some } i \in \mathcal{J}.$$

A mechanism is non-bossy (Satterthwaite and Sonnenschein, 1981) if when the mechanism gives the same utility to an individual in any two problems that only differ by this individual's utility function, then all individuals should also be indifferent. We have two versions of this property in our cardinal framework: a mechanism φ is **non-bossy** (or **utility non-bossy**) if, for every individual i, $v_i, v'_i \in \mathbf{V}_i$, and $v_{-i} \in \mathbf{V}_{-i}$,

$$\hat{v}_i(\varphi[v_i, v_{-i}]) = \hat{v}_i(\varphi[v'_i, v_{-i}]) \text{ for each } \hat{v}_i \in \{v_i, v'_i\} \implies v_j(\varphi[v_i, v_{-i}]) = v_j(\varphi[v'_i, v_{-i}]) \qquad \forall\, j \neq i.$$

In rich domains, non-bossiness is equivalent to the following notion of **outcome non-bossiness**: for every individual i, $v_i, v'_i \in \mathbf{V}_i$, and $v_{-i} \in \mathbf{V}_{-i}$,

$$\hat{v}_i(\varphi[v_i, v_{-i}]) = \hat{v}_i(\varphi[v'_i, v_{-i}]) \text{ for each } \hat{v}_i \in \{v_i, v'_i\} \implies \varphi[v_i, v_{-i}] = \varphi[v'_i, v_{-i}].$$

Beyond rich domains, outcome non-bossiness is in general a more demanding concept. Our first result relates the these three properties. The conjunction of the two non-cooperative properties: strategy-proofness and non-bossiness is equivalent to group strategy-proofness.[21]

Proposition 1 *In a rich domain, a mechanism is group strategy-proof if, and only if, it is strategy-proof and utility non-bossy.*

Restricting attention to ordinal mechanisms, Pápai (2000) proved the equivalence of these conditions in ordinal house allocation environment, and we provide a new argument which allows us to establish it for rich utility domains and any cardinal mechanism.[22]

3 Ordinal Simplicity

Our main results of this section establish that cardinal mechanisms are actually ordinal whenever they satisfy natural and commonly imposed assumptions such as group strategy-proofness. We thus provide a microfoundation for the (standard in the no-transfers literature) focus on ordinal mechanisms.

We say a mechanism φ is **ordinal** (or **utility ordinal**) if for any two utility profiles $v, v' \in \mathbf{V}$ which induce the same preference profile, for every $i \in \mathcal{I}$, we have $\hat{v}_i(\varphi[v]) = \hat{v}_i(\varphi[v'])$ for each $\hat{v}_i \in \{v_i, v'_i\}$. On rich domains this property is equivalent to the following: a mechanism φ is **outcome ordinal** if for any two utility profiles $v, v' \in \mathbf{V}$ which induce the same preference profile, for every $i \in \mathcal{I}$, we have $\varphi[v] = \varphi[v']$. Beyond rich domains, outcome ordinality is a stronger condition than utility ordinality.

An outcome ordinal mechanism only requires the ordinal information about the utility function, i.e., the underlying preferences, to determine its outcome. Accordingly, and without loss of generality though with slight abuse of notation, we sometimes treat an outcome ordinal mechanism φ as a mapping $\varphi : \mathbf{R} \to \mathcal{A}$. Many real-life mechanisms are outcome ordinal, and hence they are ordinal. For example, voting schemes for many major elections and mechanisms used in school choice throughout the world are largely ordinal, as mentioned in the Introduction.

We have two version of this implication for utility non-bossiness and ordinality and the stronger, outcome non-bossiness and ordinality.

3.1 Ordinality and Group Incentives

We start our analysis with strategy-proof mechanisms. We have the following main result:

Theorem 1 *If φ is a strategy-proof and non-bossy mechanism, then it is ordinal. In particular, if φ is a group strategy-proof mechanism, then it is ordinal.*

[21] Both of these properties are non-cooperative in the sense that they relate mechanism's outcomes under two scenarios when a single individual makes unilateral utility-revelation deviations.

[22] We prove this result in conjunction with another equivalence result in Appendix A: In a rich domain, a cardinal mechanism is group strategy-proof if, and only if, it is monotonic (Proposition 6). This latter result is in spirit similar to a result by Takamiya (2007) proven for only ordinal mechanisms.

Proof of Theorem 1. Suppose φ is a strategy-proof mechanism and non-bossy. Consider two utility profiles $v, v' \in \mathbf{V}$ that induce the same preference profile $\succcurlyeq$. Let $a = \varphi[v]$. The argument is by induction on $k \in \{1, \ldots, |\mathcal{I}|\}$ and shows that for any $\mathcal{J} \subsetneq \mathcal{I}$ such that $|\mathcal{J}| = k - 1$ we have $\hat{v}_i(\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}]) = \hat{v}_i(a)$ where $\hat{v}_i = v_i$ for each $i \in \mathcal{I} - \mathcal{J}$ and $\hat{v}_i = v'_i$ for each $i \in \mathcal{J}$.

For the base case $k = 1$ this assumption is trivially satisfied. Fix $\mathcal{J}$ such that $|\mathcal{J}| = k - 1$. Let $i \in \mathcal{I} - \mathcal{J}$. By strategy-proofness of φ, $v_i(a) = v_i(\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}]) \geqslant v_i(\varphi[v'_{\mathcal{J}\cup\{i\}}, v_{-\mathcal{J}\cup\{i\}}])$ and $v'_i(\varphi[v'_{\mathcal{J}\cup\{i\}}, v_{-\mathcal{J}\cup\{i\}}]) \geqslant v'_i(\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}]) = v'_i(a)$. Since both utilities induce the same preference $\succcurlyeq_i$, we have $a \succcurlyeq_i \varphi[v'_{\mathcal{J}\cup\{i\}}, v_{-\mathcal{J}\cup\{i\}}]$ and $\varphi[v'_{\mathcal{J}\cup\{i\}}, v_{-\mathcal{J}\cup\{i\}}] \succcurlyeq_i a$, implying $a \sim_i \varphi[v'_{\mathcal{J}\cup\{i\}}, v_{-\mathcal{J}\cup\{i\}}]$ and thus, $\hat{v}_i(\varphi[v'_{\mathcal{J}\cup\{i\}}, v_{-\mathcal{J}\cup\{i\}}]) = \hat{v}_i(a)$ for each $\hat{v}_i \in \{v_i, v'_i\}$. By non-bossiness of φ, we have $\hat{v}_j(\varphi[v'_{\mathcal{J}\cup\{i\}}, v_{-\mathcal{J}\cup\{i\}}]) = \hat{v}_j(\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}]) = \hat{v}_i(a)$ where $\hat{v}_j = v_j$ for each $j \in \mathcal{I} - \mathcal{J}$ and $\hat{v}_j = v'_j$ for each $j \in \mathcal{J}$ completing the proof of the induction and showing that φ is ordinal.

Because strategy-proof and non-bossy mechanism are group strategy-proof (directly from definitions), the just proven first claim of the theorem immediately implies its second claim. QED

An outcome-based analogue of Theorem 1 also holds true and the following obtains: *If φ is a strategy-proof and outcome non-bossy mechanism, then it is outcome ordinal.* The proof is effectively the same as the proof of Theorem 1.[23]

3.2 Ordinality and Arrovian Efficiency

The most commonly accepted efficiency concept is Pareto efficiency. An alternative is Pareto efficient for a utility profile $v \in \mathbf{V}$ if no other alternative would make everybody weakly better off and at least one individual better off; that is, an alternative $a \in \mathcal{A}$ is **Pareto efficient** if there exists no alternative $b \in \mathcal{A}$ such that for every $i \in \mathcal{I}$, $v_i(b) \geqslant v_i(a)$ and for some $j \in \mathcal{I}$, $v_j(b) > v_j(a)$. We define a mechanism to be **Pareto efficient** if it finds a Pareto-efficient alternative for every problem. Pareto efficiency is a weak efficiency requirement and, as discussed in the Introduction, Arrow criticized it for its failure to uniquely determine the best outcome; that is, for not being resolute.

We analyze social welfare criteria imposed by social welfare functions. For every utility profile, a social welfare function determines a (possibly incomplete) societal ranking of alternatives.[24] We formalize Arrovian efficiency notion by first defining Arrow's notion of a social welfare function. We denote by $\mathcal{P}^S$ the set of strict partial orderings over alternatives in $\mathcal{A}$, where a strict partial ordering is a binary relation that is anti-symmetric and transitive, but not necessarily complete. We refer to elements of $\mathcal{P}^S$ as **social rankings** and denote a generic element as $\succ^S$. A **social welfare function (SWF)** $\Phi : \mathbf{V} \to \mathcal{P}^S$ chooses a social ranking for each utility profile. An SWF Φ satisfies the **Pareto** postulate (or is **unanimous**) if: for every utility profile v and any two alternatives a and b if $v_i(a) \geqslant v_i(b)$ for every $i \in \mathcal{I}$, with at least one strict inequality, then $a\ \Phi(v)\ b$.[25] An SWF

[23]The only needed modifications in the proof are as follows: (i) the inductive assumption becomes: for $\mathcal{J} \subsetneq \mathcal{I}$ such that $|\mathcal{J}| = k - 1$ for some $k \in \{1, \ldots, |\mathcal{I}|\}$ we have $\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}] = a$, and (ii) the utility equality implied by non-bossiness of φ is replaced with $\varphi[v'_{\mathcal{J}\cup\{i\}}, v_{-\mathcal{J}\cup\{i\}}] = \varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}] = a$ implied by outcome non-bossiness of φ.

[24]For analysis of welfare with partial orderings, see e.g. see Sen (1970, 1999), Weymark (1984), and Curello and Sinander (2020).

[25]This is an extension of Arrovian Pareto postulate in the universal strict preference domain to rich domains, and is

Φ is **resolute** if, for every utility profile v there is an alternative a such that $a\ \Phi(v)\ b$ for every $b \in \mathcal{A} - \{a\}$. An SWF Φ satisfies **independence of irrelevant alternatives (IIA)** postulate if: for every $v, v' \in \mathbf{V}$ and $a, b \in \mathcal{A}$, if all individuals rank a and b in the same way under both utility profiles, i.e., for every $i \in \mathcal{I}$, $v_i(a) \geqslant v_i(b) \iff v'_i(a) \geqslant v'_i(b)$, then $a\ \Phi(v)\ b \iff a\ \Phi(v')\ b$. In particular, a and b are comparable under $\Phi(v)$ if, and only if, they are comparable under $\Phi(v')$.[26]

We say that an alternative a is **efficient with respect to social ranking** $\succ^S \in \mathcal{P}^S$ if it maximizes the social welfare, that is $a \succ^S b$ for every $b \in \mathcal{A} - \{a\}$. A mechanism φ is **efficient with respect to an SWF** Φ if for any profile of individuals' utilities v, the alternative $\varphi[v]$ is efficient with respect to $\Phi(v)$. A mechanism is **Arrovian efficient** if it is efficient with respect to some SWF that satisfies the Pareto, resoluteness, and IIA postulates. Proposition 2 in the next subsection shows how one can check Arrovian efficiency without the need to construct SWF.

Arrovian efficiency implies ordinality:

Theorem 2 *If φ is an Arrovian-efficient mechanism, then it is ordinal and outcome ordinal.*

On the other hand, a Pareto-efficient mechanisms, even strategy-proof and Pareto-efficient mechanisms, need not be ordinal as the following example illustrates.

Example 1: Consider a house allocation environment with three agents and three houses, and the following mechanism: Agent 1 is assigned her top choice. If the reported utility of Agent 1 for her top choice is above 50 utils, then Agent 2 is assigned her top choice among the two remaining best houses, and otherwise Agent 3 is. The last agent, in either case, is assigned the remaining house. This mechanism is strategy-proof and Pareto efficient, but not ordinal. ■

The above proposition is a direct implication of resoluteness and IIA properties of the underlying SWFs.

Proof of Theorem 2. Take any two utility profiles $v, v' \in \mathbf{V}$ that induce the same preference profile $\succcurlyeq$. Let Φ be the Arrovian SWF with respect to which φ is efficient. Suppose to the contrary of the claim $a = \varphi[v] \neq \varphi[v'] = b$. Then, as Φ is resolute, $\Phi(v)$ and $\Phi(v')$ both compare a and b. Moreover, v and v' satisfy Hypothesis 1 in the definition of the IIA property. Then by the IIA property of Φ, as $a = \varphi[v]\ \Phi(v)\ b$ we have $a\ \Phi(v')\ b$. This conclusion contradicts Φ is the Arrovian SWF with respect to which φ is efficient as $b = \varphi[v']$. QED

We allow both incomplete and complete Arrovian SWFs. Arrow (1963) motivated looking at the complete social rankings over outcomes for the environments by the need to accommodate a variety of resource constraints. A complete social ranking can be used to determine the socially optimal decision for any constrained social welfare maximization problem. As an application

also known as the strong Pareto postulate.

[26] We can weaken the Pareto postulate by restricting it to alternatives that are are comparable by $\Phi(v)$. We can also weaken IIA to alternatives that are comparable under both $\Phi(v)$ and $\Phi(v')$. With either or both of these weakenings, all our results go through with basically the same arguments.

of this concept, in the common ownership economies with unit demand—the house allocation problem—we characterize the full class of strategy-proof cardinal mechanisms that are efficient with respect to a complete Arrovian social welfare function (see Section 4.2). They turn out to be a class that we refer to as almost sequential dictatorships.

3.3 Ordinality and Auditability

One can check that a mechanism is Arrovian efficient looking at the mechanisms directly, without a need to construct Arrovian social welfare functions. The verification is anchored in the following property of mechanisms: A mechanism φ is **simply auditable** (or, **auditable**, for brevity) if, whenever $\varphi[v] \neq \varphi[v']$ for any two utility profiles $v, v' \in \mathbf{V}$ then there is an agent i that ranks these two alternatives differently under v_i and v'_i, that is either $v'_i(\varphi[v]) \geqslant v'_i(\varphi[v'])$ and $v_i(\varphi[v]) < v_i(\varphi[v'])$, or else $v'_i(\varphi[v]) < v'_i(\varphi[v'])$ and $v'_i(\varphi[v]) \geqslant v'_i(\varphi[v'])$. This concept captures the idea that an auditor (or a court)—that knows the mapping φ but does not know what profile of utilities was reported—can falsify the proposed outcome of the mechanism ($\varphi[v]$) by verifying just one (judiciously chosen) agent's comparison between this outcome and just one (judiciously chosen) challenger alternative ($\varphi[v']$).[27] The falsification hence requires documenting just one comparison. In this sense the falsification imposes minimal verification cost. The auditor of course would need to know which challenger alternative and which agent's comparison to focus on. As discussed in introduction knowing where the problem lies is often much less costly than formally verifying the problem.

Auditability is related to both Arrow's IIA and non-bossiness. The main substantive difference between auditability and IIA is that IIA is a normative postulate imposed by Arrow on preference aggregation, while auditability captures whether the outcome of the mechanism can be easily falsified. [28] Despite this difference, on rich domains the following equivalence allows one to check Arrovian efficiency of a mechanism:

Proposition 2 *Suppose the utility domain is rich. Then a mechanism is Arrovian efficient if, and only if, it is Pareto efficient and auditable.*

Both auditability is a property of mechanisms, and, as the next result shows, it is weaker than group strategy-proofness but stronger than non-bossiness:

Proposition 3 *Suppose the utility domain is rich. Any group strategy-proof cardinal mechanism is auditable, and any auditable mechanism is non-bossy. Neither of the converse implications holds true, even for Pareto-efficient mechanisms.*

[27] The assumption that proposed outcome takes the form $\varphi[v]$ for some utility profile v is without loss because otherwise the auditor could infer that the proposed outcome is false merely from the knowledge of φ.

[28] Also auditability is technically weaker as one needs the ranked alternatives underlying the Arrovian social welfare function, but not ranking pertaining to top alternatives under two different .

We prove the above two propositions in Appendix E.1. The logical relations between auditability and Arrovian efficiency and non-bossiness allow us to obtain the following corollary from Theorems 1 and 2:

Theorem 3 *In any rich utility domain:*

- *any Pareto-efficient and auditable cardinal mechanism is ordinal and outcome ordinal;*
- *any strategy-proof and auditable cardinal mechanism is ordinal and outcome ordinal.*

In the first part of Theorem 3 we cannot weaken auditability to non-bossiness.

4 Applications: Characterizations of Mechanism Classes in Rich Domains

In our applications, we use the following equivalence which follows directly from Propositions 1, 2, 3:

Proposition 4 *Suppose the utility domain is rich. A cardinal mechanism is strategy-proof and Arrovian efficient if, and only if, it is strategy-proof, auditable, and Pareto efficient if, and only if, it is group strategy-proof and Pareto efficient.*

4.1 Strategy-Proof Public Choice

The most straightforward application of our concepts is in the universal strict preference domain, which is also called the universal public choice environment. This environment consists of all strict preference relations over alternatives, each of which can be interpreted as a candidate in an election. In the universal domain, by Proposition 3 and the example in its proof and the fact that every mechanism is non-bossy, we obtain the following result:

Proposition 5 *In the universal strict-preference cardinal domain, a strategy-proof cardinal mechanism is auditable. But the converse is not necessarily true, even for Pareto-efficient ordinal mechanisms.*

Taken together with Proposition 2, it implies the following:

Corollary 1 *In the universal strict preference domain, for a strategy-proof cardinal mechanism the following two conditions are equivalent:*

- *Pareto efficiency,*
- *Arrovian efficiency.*

In Appendix A, we provide some further results relevant for public choice environments and in particular we extend Muller and Satterthwaite's (1977) characterization of strategy-proofness to the cardinal domain.

4.2 Strategy-Proof Allocation with Single-Unit Demand

Formally, a **house allocation** environment (Hylland and Zeckhauser, 1979) consists of the set of individuals $\mathcal{I}$ and a set of **houses** H. A social alternative for this problem is a matching. To simplify the definition of a matching, we focus on environments in which $|H| \geqslant |\mathcal{I}|$. To define a matching, let us start with a more general concept that we use frequently below. A submatching is an allocation of a subset of houses to a subset of individuals, such that no two different individuals get the same house. Formally, a **submatching** is a one-to-one function $s : \mathcal{J} \to H$; where for $\mathcal{J} \subseteq \mathcal{I}$, using the standard function notation, we denote by $s(i)$ the assignment of individual $i \in \mathcal{J}$ under s, and by $s^{-1}(h)$ the individual that got house $h \in s(\mathcal{J})$ under s. Let $\mathcal{S}$ be the set of submatchings. For every $s \in \mathcal{S}$, let $\mathcal{I}_s$ denote the set of individuals matched by s and $H_s \subseteq H$ denote the set of houses matched by s. For every $h \in H$, let $\mathcal{S}_{-h} \subset \mathcal{S}$ be the set of submatchings $s \in \mathcal{S}$ such that $h \in H - H_s$, i.e., the set of submatchings at which house h is unmatched. By virtue of the set-theoretic interpretation of functions, submatchings are sets of individual-house pairs and are ordered by inclusion. A **matching** is a maximal submatching; that is, $a \in \mathcal{S}$ is a matching if $\mathcal{I}_a = \mathcal{I}$. As before, let $\mathcal{A} \subset \mathcal{S}$ be the set of matchings. We will write $\overline{\mathcal{I}_s}$ for $\mathcal{I} - \mathcal{I}_s$ and $\overline{H_s}$ for $H - H_s$ for short. We will also write $\overline{\mathcal{A}}$ for $\mathcal{S} - \mathcal{A}$. Each individual $i \in \mathcal{I}$ has a preference ranking $\succcurlyeq_i \in \mathbf{R}_i$ over houses. As each mechanism φ we introduce in this section preserves ordinality, we keep the notation $\varphi[\succcurlyeq]$ to denote the mechanism outcome.

Pycia and Ünver (2017) constructed the class of Trading Cycle mechanisms and showed that this is exactly the class of strategy-proof, non-bossy, and Pareto-efficient ordinal mechanisms. This mechanism class is defined through an iterative algorithm, which matches some individuals in every round. In each round, each object is controlled by a not-yet-matched individual; the control can take the form of ownership (which corresponds to the ownership in Gale's Top Trading Cycles and in Pápai (2000) Hierarchical Exchange and allows the owner to trade it or be matched with it directly) or of brokerage (which allows the individual to trade the object but not to be matched with it directly).

Pycia and Ünver (2017) define a control-rights structure as a function of the submatching that is fixed: A **structure of control rights** is a collection of mappings

$$(\kappa, \beta) = \left\{(\kappa_s, \beta_s) : \overline{H_s} \to \overline{\mathcal{I}_s} \times \{\text{ownership, brokerage}\}\right\}_{s \in \overline{\mathcal{A}}}.$$

The functions κ_s of the control-rights structure tell us which unmatched individual controls any particular unmatched house at a submatching s, where **at** s is the terminology we use when some individuals and houses are already matched with respect to s. Agent i **controls** house $h \in \overline{H_s}$ at submatching s when $\kappa_s(h) = i$. The type of control is determined by functions β_s. We say that the individual $\kappa_s(h)$ **owns** h at s if $\beta_s(h)$ =ownership, and that the individual $\kappa_s(h)$ **brokers** h at s if $\beta_s(h)$ =brokerage. In the former case, we call the individual an **owner** and the controlled house an **owned house**. In the latter case, we use the terms **broker** and **brokered house**. Notice that each controlled (owned or brokered) house is unmatched at s, and any unmatched house is controlled by some uniquely determined unmatched individual. Pycia and Ünver (2017) construct **trading-cycles (TC)** mechanisms $\psi^{\kappa,\beta}$ for control-rights structures (κ, β) that satisfy consistency conditions that they introduce. As they show these consistence conditions are needed to guarantee that the induced mechanisms are well-defined, strategy-proof, non-bossy, and Pareto efficient. For completeness we describe their consistency conditions and their TC algorithm—that constructs $\psi^{\kappa,\beta}$ for any consistent control-rights structure (κ, β)—-in Appendix D.

One useful feature of the TC mechanisms is that we can, without loss of generality, rule out the existence of brokers at some submatching s if there is a single owner at s. Following Pycia and Ünver (2017), we formalize this property as follows:

Remark 1 *Pycia and Ünver (2017) For every TC mechanism such that for some s there is only one owner and one broker, there is an equivalent TC mechanism such that at s there are no brokers and the same owner owns all houses.*

Using Theorem 1, Proposition 4, and Pycia and Ünver (2017)'s characterization we obtain the following corollary:

Theorem 4 *In the unit-demand house allocation environment with strict preferences, a cardinal mechanism is strategy-proof and Arrovian efficient if, and only if, it is group strategy-proof and Pareto efficient if, and only if, it is a TC mechanism.*

So far, we allowed welfare functions to incompletely rank social outcomes. We now show that a class that we refer to as almost sequential dictatorships is exactly the mechanisms that are strategy-proof and Arrovian efficient with respect to complete SWF, that is SWF that always rank all outcomes. As mentioned before complete Arrovian SWFs are important when there are uncertain resource constraints, and the society wants to come up with a well-defined welfare function for choosing a social alternative regardless of the realized resource constraints. Thus what kind of cardinal strategy-proof mechanisms reach to this goal if we insist on Arrovian efficiency with respect to a complete Arrovian SWF?

We leverage our main results to answer this question. First we define the following class: a **top-trading-cycles** (**TTC**) (or **hierarchical exchange**) mechanism is a TC mechanism with a control-rights structure in which no house is ever brokered at any submatching (Pápai, 2000). A TTC mechanism $\psi^{\kappa,\beta}$ will be denoted by dropping β from its notation as ψ^{κ}. TTC mechanisms form a strict subclass of TC mechanisms. Let us start with a lemma showing that not every TTC is Arrovian efficient with respect to a complete SWF.[29]

Lemma 1 *Suppose that $|H| \geqslant |\mathcal{I}| = 2$ and a TTC mechanism is Arrovian efficient with respect to a complete SWF. Then in this mechanism no individual can own two houses while a second individual owns a house.*

[29] All proofs of the results in this subsection are in E.2.

We will use this lemma to characterize strategy-proof and Arrovian efficient mechanisms for $|H| \geqslant |\mathcal{I}|$. If $|H| > |\mathcal{I}|$ then the resulting class consists of sequential dictatorships. Formally, a **sequential dictatorship** is a TTC mechanism ψ^κ such that for every $s \in \overline{\mathcal{A}}$ and $h, h' \in \overline{H_s}$, $\kappa_h(s) = \kappa_{h'}(s)$, i.e., an unmatched individual owns all unmatched houses at s. For notational convenience, we will represent each $\kappa_h(\cdot)$ as $\kappa(\cdot)$. Then, the initial owner $\kappa(\emptyset)$ is assigned her top choice house; subject to this fixed submatching, say s, the owner of all remaining houses at s, $\kappa(s)$ is assigned her top choice among these houses. Let s' be the fixed submatching so far. New owner $\kappa(s')$ is assigned his top choice among remaining houses, so on. Sequential dictatorships turn out to be the class of Arrovian-efficient and strategy-proof mechanisms for this case:

Theorem 5 *In the unit demand house allocation environment with strict preferences, suppose $|H| > |\mathcal{I}|$. A cardinal mechanism is strategy-proof and Arrovian efficient with respect to a complete social welfare function if, and only if, it is a sequential dictatorship.*

The above simple statement of the theorem relies on there being more houses than agents. A larger class of mechanisms is strategy-proof and Arrovian efficient with respect to a complete social welfare function when $|H| = |\mathcal{I}|$. An **almost sequential dictatorship** is a TTC mechanism ψ^κ such that for every $s \in \overline{A}$ such that $|\overline{H_s}| \neq 2$ we have $\kappa_h(s) = \kappa_{h'}(s)$ for every $h, h' \in \overline{H_s}$. Note that the only mechanisms that are not sequential dictatorships in this class are mechanisms that assign to different owners each of the houses when only two houses are left, but otherwise a single individual owns all houses.[30]

Theorem 6 *In the unit-demand house allocation environment with strict preferences, a cardinal mechanism is strategy-proof and Arrovian efficient with respect to a complete SWF if, and only if, it is an almost sequential dictatorship.*

We prove both above theorems in Appendix E.2, relying on Lemma 1 and two further lemmas showing that three individuals each cannot simultaneously control a house under a TC mechanism that is efficient with respect to a complete SWF.

4.3 Strategy-proof Allocation with Multi-Unit Demand

What happens if an agent could consume multiple objects instead of unit objects? When agents can have any strict utility functions over groups of houses, this domain is also a rich domain. For ordinal preference domains, Hatfield (2009) proved that all Pareto-efficient and group strategy-proof ordinal mechanisms are sequential dictatorships. With multi-unit demands, the definition of a matching and a submatching is amended so that an agent can be matched with multiple houses. We allow arbitrary strict utility functions for agents over groups of houses. In this case,

[30]Thus, when$|\mathcal{I}| = |H|$ and when two houses remain, one can be owned by one agent and the other by the other remaining agent. If a unanimous outcome exists, i.e., they prefer different houses, they are assigned their preferred house; if both agents prefer the same house, then this house's owner gets it, and the other agent gets the final remaining house.

the definition of a sequential dictatorships have to be modified so that the agent who owns all houses at a submatching, is assigned her best house bundle. We then obtain the following result as a corollary to Proposition 4, Theorem 1, and Hatfield's result:

Theorem 7 *In the multi-demand house allocation environment with strict preferences, a cardinal mechanism is strategy-proof and Arrovian efficient if, and only if, it is group strategy-proof and Pareto efficient if, and only if, it is a sequential dictatorship.*

5 Conclusion

We provided a micro foundation for the common focus on ordinal mechanisms in market design without transfers. In the process, we brought the Arrow's efficiency program to market design. Looking at a large class of rich utility domains, we established equivalences between group strategy-proofness and the conjunction of individual strategy-proofness and non-bossiness, as well as between other useful mechanism design concepts. We leveraged these results to characterize what classes of allocation mechanisms satisfy natural incentive-compatibility, efficiency, and auditability concepts.

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A Monotonicity and Strategy-Proofness

Given a mechanism φ, a utility profile v' is a **φ-monotonic transformation** of another profile v if, for every individual $i \in \mathcal{I}$ and alternative $a \in \mathcal{A}$, $v_i(\varphi[v]) \geqslant v_i(a)$ implies $v'_i(\varphi[v]) \geqslant v'_i(a)$. Thus, for every individual, the set of alternatives weakly better than the mechanism's outcome under the base profile v weakly shrinks when we go from the base profile to its monotonic transformation v'. A mechanism φ is (**Maskin**) **monotonic** (Maskin, 1999) if, for every $v \in \mathbf{V}$, $\varphi[v'] = \varphi[v]$ for every $v' \in \mathbf{V}$ that is a φ-monotonic transformation of v.

Restricting attention to ordinal mechanisms, Takamiya (2001, 2007) proved the equivalence of monotonicity and group strategy-proofness. We show that the restriction to ordinal mechanisms can be relaxed:

Proposition 6 *In a rich utility domain, a cardinal mechanism is monotonic if, and only if, it is group strategy-proof.*

We prove this proposition in conjunction with Proposition 1.

Proof of Propositions 1 and 6. (Group strategy-proofness $\implies$ strategy-proofness and non-bossiness) By definition, any group strategy-proof mechanism is immune to all single-person group deviations, and hence, it is also strategy-proof. To the contrary to the claim, suppose a group strategy-proof mechanism φ is not non-bossy. Then there exists some individual i, a utility profile $v \in \mathbf{V}$, and i's utility $v_i' \in \mathbf{V}_i$ such that $u_i(\varphi[v]) = u_i(\varphi[v_i', v_{-i}])$ for each $u_i \in \{v_i, v_i'\}$ and yet there exists some individual $j \neq i$ such that $v_j(\varphi[v]) \neq v_j(\varphi[v_i', v_{-i}])$. Consider the group $\mathcal{J} = \{i, j\}$. Suppose without loss of generality, $v_j(\varphi[v]) > v_j(\varphi[v_i', v_{-i}])$. Then consider the group deviation (v_i, v_j) from the profile (v_i', v_{-i}): individual i is indifferent while individual j is better off contradicting group strategy-proofness of φ. Thus, we showed that φ is also non-bossy.

(Under richness: Individual strategy-proofness and non-bossiness $\implies$ monotonicity) Consider a rich utility domain $\mathbf{V}$. Let φ be an strategy-proof and non-bossy mechanism. Consider a utility profile $\mathbf{V}$. Let $v' \in \mathbf{V}$ be one of its φ-monotonic transformations. We prove this part by induction. Suppose as the inductive assumption, we proved that for a given $\mathcal{J} \subset \mathcal{I}$ (for the base case $\mathcal{J} = \emptyset$ trivially holds), we showed that $\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}] \equiv_j \varphi[v]$ for every $j \in \mathcal{I}$. Consider an individual $i \in \mathcal{I} - \mathcal{J}$. Let $\tilde{\mathcal{J}} = \{i\} \cup \mathcal{J}$. First we establish that $\varphi[v'_{\tilde{\mathcal{J}}}, v_{-\tilde{\mathcal{J}}}] \equiv_i \varphi[v]$: Suppose not, to the contrary of the claim. Let $a' = \varphi[v'_{\tilde{\mathcal{J}}}, v_{-\tilde{\mathcal{J}}}] \not\equiv_i \varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}] = a \equiv_i \varphi[\succcurlyeq]$. If $v_i'(a') > v_i'(a)$, then $v_i(a') > v_i(a)$ by choice of v_i' as part of monotonic transformation v', and in turn, this contradicts strategy-proofness of φ for i, as she can report v_i' and be better off while her utility is v_i and others have $(v'_{\mathcal{J}}, v_{-\tilde{\mathcal{J}}})$. On the other hand, if $v_i'(a) > v_i'(a')$, this contradicts strategy-proofness of φ for i, too, as she can report v_i and be better off while her tru utility is v_i' and others have $(v'_{\mathcal{J}}, v_{-\tilde{\mathcal{J}}})$. Thus, $v_i'(a) = v_i'(a')$. Since $a \not\equiv_i a'$, this last statement contradicts part 2 of the richness assumption. Thus $a \equiv_i a'$. Then non-bossiness of φ implies that $a \equiv_j a'$ for every $j \in \mathcal{I}$. Inductive assumption implies that $\varphi[v'_{\tilde{\mathcal{J}}}, v_{-\tilde{\mathcal{J}}}] \equiv_j \varphi[v_{-\tilde{\mathcal{J}}}]$ for every $j \in \mathcal{I}$.

(Under richness: Monotonicity $\implies$ group-strategy-proofness). Consider a rich domain $\mathbf{V}$. Let φ be a monotonic mechanism. Consider a utility profile $v \in \mathbf{V}$, a group $\mathcal{J} \subseteq \mathcal{I}$, and a possible deviation $v'_{\mathcal{J}} \in \mathbf{V}_{\mathcal{J}}$. Suppose $v_j'(\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}]) \geqslant v_j'(\varphi[\succcurlyeq])$ for every $j \in \mathcal{J}$ and for some individual $i \in \mathcal{J}$ the inequality is strict. Consider the utility profile of $\mathcal{J}$, $v^*_{\mathcal{J}} \in \mathbf{V}_{\mathcal{J}}$ such that a' gives higher utility than a and every other equivalence class of alternatives are valued below these two alternatives' equivalence classes. Then $(v^*_{\mathcal{J}}, v_{-\mathcal{J}}) \in \mathbf{V}$ and it is a φ- monotonic transformation of v, and hence, $\varphi[v^*_{\mathcal{J}}, v_{-\mathcal{J}}] \equiv_j a$ for al $j \in \mathcal{I}$ by monotonicity of φ. Since a' is the top alternative in v_j^* for every $j \in \mathcal{J}$ and $\varphi[v'_{\mathcal{J}}, v_{-\mathcal{J}}] = a'$, $(v^*_{\mathcal{J}}, v_{-\mathcal{J}})$ is also a φ-monotonic transformation of $(v'_{\mathcal{J}}, v_{-\mathcal{J}})$, and hence, $\varphi[v^*_{\mathcal{J}}, v_{-\mathcal{J}}] \equiv_j a'$ for every $j \in \mathcal{I}$ by monotonicity of φ. Since $a \not\equiv_i a'$, we obtain a contradiction. Thus, φ is group strategy-proof. QED

In the universal ordinal preference domain, Muller and Satterthwaite (1977) show that monotonicity is equivalent to strategy-proofness. Because every mechanism is non-bossy in the universal utility domain, our Proposition 6 and Theorem 1 imply that Muller and Sattherwaite's equivalence extends to cardinal mechanisms:

Corollary 2 *In the universal utility domain, a cardinal mechanism is strategy-proof if, and only if, it Maskin monotonicity.*

B Illustrating Arrovian Efficiency

To illustrate the results and our concepts, let us look at the this with three individuals 1, 2, and 3, three houses A, B, and C, and no outside options. We will use ordinal preferences in this construction, but it can be generalized to cardinal utility profiles as well. Given an alternative, which is a matching of houses to individuals, a let $a(i)$ refer to the house assigned to an individual i under a. Individuals' preferences are denoted by strict preferences over houses, by slight abuse of notation, instead of alternatives and their equivalence relations are over matchings that match them with the same house. Here, let us consider three examples of mechanisms illustrating the conditions we study.

Example 2: With three individuals $1, 2, 3$ and three houses A, B, C (thus, with 6 alternatives) the serial dictatorship φ in which individual 1 chooses first the house she would like to receive and individual 2 chooses second is well-known to be strategy-proof, non-bossy, and Pareto efficient, as well as group strategy-proof and monotonic.

It is straightforward to see that this serial dictatorship is Arrovian efficient with respect to the following SWF: a is ranked above b if, and only if, (a) 1 prefers a to b, or (b) 1 is indifferent and 2 prefers a to b.

As φ treats all objects in a symmetric (neutral) way, to establish the serial dictatorship's auditability, it is sufficient to look at a preference profile $\succcurlyeq$ such that $\varphi[\succcurlyeq] = \{(1, A), (2, B), (3, C)\}$, a different alternative b and any preference profile $\succcurlyeq'$ such that $\succcurlyeq'_i$ keeps the same ranking as $\succcurlyeq_i$ between $\varphi[\succcurlyeq]$ and b for every individual i and to show that $\varphi[\succcurlyeq'] \neq b$. To verify this inequality consider two cases:

- $A \neq b(1)$. Then $A \succ_1 b(1)$ because 1 being the first dictator chose her top choice under $\succcurlyeq_i$. Hence, $A \succ'_1 b(1)$. 1 is not choosing $b(1)$ when having preference ranking $\succ'_1$ and thus $\varphi[\succcurlyeq'] \neq b$.
- $A = b(1)$. Then either :
  - ⋆ $B \neq b(2)$. Then, $B \succ_2 b(2)$ by an argument similar to the previous case. If $\varphi[\succcurlyeq'](1) = B$ then $\varphi[\succcurlyeq'] \neq b$, and the auditability inequality obtains. If $\varphi[\succcurlyeq'](1) \neq B$ then either $\varphi[\succcurlyeq'](1) \neq A = b(1)$ and the auditability inequality obtains, or $\varphi[\succcurlyeq'](1) = A = b(1)$ and hence B is available when 2's assignment is determined, and thus, $\varphi[\succcurlyeq'] \succcurlyeq'_2 B \succ'_2 b(2)$, and hence, $\varphi[\succcurlyeq'] \neq b$ and the auditability inequality obtains.
  - ⋆ $B = b(2)$. Then $C = b(3)$ contrary to $b \neq \varphi[\succcurlyeq]$.

■

Example 3: We now modify the serial dictatorship of the previous example and consider mechanisms ψ in which 1 chooses first; then 2 chooses second if 1 prefers B over C, else 3 chooses second. This mechanism is an example of a ranking-dependent sequential dictatorship, and is also strategy-proof and Pareto efficient. However, mechanism ψ is neither Arrovian efficient nor non-bossy nor auditable. To see the latter three points, let us look at the following two preference profiles, which differ only in how individual 1 ranks objects:

$$\succcurlyeq = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline A & A & A \\ B & B & B \\ C & C & C \end{array} \qquad \succcurlyeq' = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline A & A & A \\ C & B & B \\ B & C & C \end{array},$$

and notice that

$$\begin{aligned} \psi[\succcurlyeq] &= \{(1,A),(2,B),(3,C)\}, \\ \psi[\succcurlyeq'] &= \{(1,A),(2,C),(3,B)\}. \end{aligned}$$

Mechanism ψ does not satisfy non-bossiness because from $\succcurlyeq$ to $\succcurlyeq'$ only 1's preference changes and her assignment does not change, and yet other individuals' assignments change (leading to different equivalence classes of alternatives for either individual 2 and 3).

Mechanism ψ does not satisfy Arrovian efficiency. Indeed, by way of contradiction assume that ψ is Arrovian efficient with respect to some Arrovian SWF Ψ. Then $\Psi(\succcurlyeq)$ ranks alternative $\psi[\succcurlyeq]$ above $\psi[\succcurlyeq']$, and $\Psi(\succcurlyeq')$ ranks $\psi[\succcurlyeq']$ above $\psi[\succcurlyeq]$. But, this violates IIA, a contradiction that shows that ψ is not Arrovian efficient.

Mechanism ψ does not satisfy auditability as we can contest the alternative $\psi[\succcurlyeq]$ with alternative $b = \psi[\succcurlyeq']$.

Mechanism ψ does not satisfy group strategy-proofness because the group $\{1,3\}$ can beneficially manipulate by reporting $\succcurlyeq'_{\{1,3\}}$ instead of $\succcurlyeq_{\{1,3\}}$ (noticing $\succcurlyeq_2 = \succcurlyeq'_2$), making individual 3 strictly better off while leaving individual 1 indifferent.

Finally, mechanism ψ does not satisfy monotonicity as $\succcurlyeq'$ is a ψ-monotonic transformation of $\succcurlyeq$ and yet the mechanism's respective outcomes are in different equivalence classes for individuals 2 and 3. ■

The following example illustrates an incomplete Arrovian SWF.

Example 4: Consider a society (or an employer) assigning one task to each of three employees. All the tasks need to be completed, and the society would like to respect the preferences of the employees in assigning the tasks as much as possible. As a second order concern, the society would like to avoid assigning Task A to employee 1 (e.g. because of a belief that employee 1 is not very good in doing this job). The society thus has an SWF that has the maximum at a

Pareto-efficient matching that does not assign Task A to employee 1 if there exists at least one Pareto-efficient matching that does not assign Task A to employee 1.

The society's SWF can be equivalently described in terms of a Trading Cycles mechanism ψ in which employee 1 brokers A, employee 2 has ownership of B and employee 3 has ownership of C: for any preference profile $\succcurlyeq_{\{1,2,3\}}$, the SWF $\Psi(\succcurlyeq)$ ranks any two distinct matchings a and b if, and only if, $a = \psi[\succcurlyeq]$ or a Pareto dominates b; the social ranking is then $a \; \Psi(\succcurlyeq) \; b$.

For instance, for the preference profile

$$\succcurlyeq = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline A & A & B \\ B & B & C \\ C & C & A \end{array},$$

the outcome of Trading Cycles ψ is $\psi[\succcurlyeq] = \{(1,B), (2,A), (3,C)\}$, and the ranking of the matchings with respect to $\Psi(\succcurlyeq)$ is given in Figure 1.

■

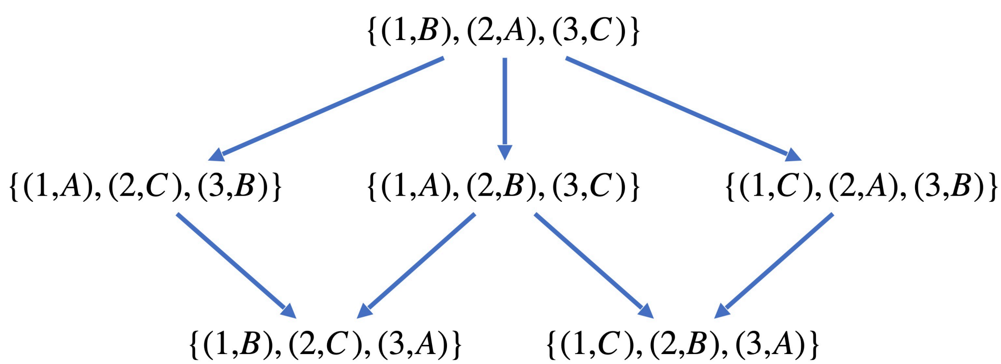

Figure 1: $\Psi(\succcurlyeq)$ in Example 4. For matching a, b, we have $a \; \Psi(\succcurlyeq) \; b$ if, and only if, there is a directed path from a to b in this graph.

C Further Equivalences and Non-Equivalences

For rich domains, many equivalences follow directly Propositions 1, 2, 3, and 6. For instance, a cardinal mechanism is Pareto efficient and group strategy-proof if, and only if, it is Pareto efficient and monotonic. Furthermore, for every strategy-proof and Pareto efficient cardinal mechanism the following five conditions are equivalent: Arrovian efficiency, auditability, group strategy-proofness, non-bossiness, and monotonicity.

At the same time, Arrovian efficiency does not in general imply individual strategy-proofness, even in the universal preference domain (see the proof of Proposition 3). One can also easily find examples that strategy-proofness does not imply Arrovian efficiency (or auditability).

D TC Algorithm and Consistent Control-Rights Structures

A structure of control rights (κ, β) is **consistent** if the following within-round and across-round requirements are satisfied for every $s \in \overline{\mathcal{A}}$:

Within-Round Requirements:

(**R1**) There is at most one brokered house at s, or $|\overline{H_s}| = 3$ and all remaining houses are brokered.

(**R2**) If i is the only unmatched individual at s, then i owns all unmatched houses at s.

(**R3**) If individual i brokers a house at s, then i does not control any other houses at s.

Across-Round Requirements: Consider submatching s' such that $s \subset s' \in \overline{\mathcal{A}}$, and an individual $i \in \overline{\mathcal{I}_{s'}}$ that owns a house $h \in \overline{H_{s'}}$ at s. Then:

(**R4**) Agent i owns h at s'.

(**R5**) If i' brokers house h' at s, and $i' \in \overline{\mathcal{I}_{s'}}$, $h' \in \overline{H_{s'}}$, then either i' brokers h' at s', or i owns h' at s'. (Notice that the latter case can only happen if i is the only individual in $\overline{\mathcal{I}_{s'}}$ who owns a house at s.)

(R6) If individual $i' \in \overline{\mathcal{I}_{s'}}$ controls $h' \in \overline{H_{s'}}$ at s, then i' owns h at $s \cup \{(i, h')\}$.

Each consistent control-rights structure (κ, β) induces a **trading-cycles (TC)** mechanism $\psi^{\kappa,\beta}$, and given a problem $\succcurlyeq \in \mathbf{R}$, the outcome matching $\psi^{\kappa,\beta}[\succcurlyeq]$ is found as follows:

Trading cycles (TC) algorithm:

The algorithm starts with empty submatching $s^0 = \varnothing$ and in each round $r = 1, 2, ...$ it matches some individuals with houses. By s^{r-1}, we denote the submatching of individuals matched before round r. If $s^{r-1} \in \overline{\mathcal{A}}$, then the algorithm proceeds with the following three steps of round r:

Step 1 *Pointing.* Each house $h \in \overline{H_{s^{r-1}}}$ points to the individual who controls it at s^{r-1}. Each individual $i \in \overline{\mathcal{I}_{s^{r-1}}}$ points to her most preferred outcome in $\overline{H_{s^{r-1}}}$.

Step 2(a) *Matching Simple Trading Cycles.* A cycle

$$h^1 \to i^1 \to h^2 \to ...h^n \to i^n \to h^1,$$

in which $n \in \{1, 2, ...\}$ and individuals $i^\ell \in \overline{\mathcal{I}_{s^{r-1}}}$ point to houses $h^{\ell+1} \in \overline{H_{s^{r-1}}}$ and houses h^ℓ point to individuals i^ℓ (here $\ell = 1, ..., n$ and superscripts are added modulo n), is simple when at least one individual in the cycle is an owner. Each individual in each simple trading cycle is matched with the house she is pointing to.

Step 2(b) *Forcing Brokers to Downgrade Their Pointing.* If there are no simple trading cycles in the preceding Step 2(a), and only then we proceed as follows (otherwise we proceed to step 3).

- ⋆ If there is a cycle in which a broker i points to a brokered house, and there is another broker or owner that points to this house, then we force broker i to point to her next choice and we return to Step 2(a).[31]
- ⋆ Otherwise, we clear all trading cycles by matching each individual in each cycle with the house she is pointing to.

Step 3 Submatching s^r is defined as the union of s^{r-1} and the set of newly matched individual-house pairs. When all individuals or all houses are matched under s^r, then the algorithm terminates and gives matching s^r as its outcome.

E Omitted Proofs

E.1 Proofs of Results in Section 3.3

Proof of Proposition 2.

(Arrovian efficiency $\implies$ Pareto efficiency) Consider an Arrovian efficient mechanism φ with respect to some SWF Φ. Suppose that for some $v \in \mathbf{V}$, $\varphi[v]$ is not Pareto efficient. Then there exists some $a \in \mathcal{A} - \{\varphi[v]\}$ such that $v_i(a) \geqslant v_i(\varphi[v])$ for every i, with a strict preference for at least one individual. Because Φ satisfies the Pareto postulate, we have $\varphi[v]\ \neg\Phi(v)\ a$,[32] which contradicts the assumption that φ is Arrovian efficient with respect to Φ.

(Arrovian efficiency $\implies$ auditability). An inspection of the definitions shows that Arrovian efficiency directly implies auditability; indeed, auditability is effectively IIA restricted to comparisons involving the top equivalence class.

(Pareto efficiency and auditability $\implies$ Arrovian efficiency). Consider a Pareto-efficient and auditable mechanism φ. We define an SWF Φ as follows: for any utility profile v and any two alternatives a and $a' \neq a$, alternative a is ranked by $\Phi(v)$ above a' if, and only if, either (i) we have $a = \varphi[v]$ or (ii) for every individual i, we have $v_i(a) \geqslant v_i(a')$ and at least for one individual i the inequality is strict (which we refer to, by sight abuse of terminology, as "individuals unanimously rank a over a'" throughout the proof). Note that Pareto efficiency of φ implies that conditions (i) and (ii) are consistent with each other, and hence, that the SWF Φ is well defined.

[31] Importantly, broker i is unique by R1.

[32] We use $\neg$ is to negate a logical statement when used in front of a binary relation. In this case its use means "$\varphi[v]\ \Phi(v)\ a$ is not true".

By definition, Φ satisfies the Pareto postulate. Furthermore, Φ is transitive: if $\Phi(v)$ ranks a^1 above a^2 and it ranks a^2 above a^3 , then it ranks a^1 above a^3. To see this: if one of these a^ℓ (for $\ell = 1, 2, 3$) equals $\varphi[v]$, then it must be that $a^1 = \varphi[v]$, and the claim is proven. If none of the a^ℓ equals $\varphi[v]$, then individuals unanimously rank a^1 above a^2 and unanimously rank a^2 above a^3; we conclude that individuals unanimously rank a^1 above a^3, and thus, $\Phi(v)$ ranks a^1 above a^3 by construction.

It remains to check that Φ satisfies IIA. Take two utility profiles v^1 and v^2 such that each individual ranks two alternatives, say a and a', in the same way under the two profiles. If the two alternatives are comparable under both $\Phi\left(v^1\right)$ then one of the following cases obtains:

Case 1: One of the alternatives is unanimously preferred to the other under v^1; then the same unanimous preference obtains under v^2 and by the construction of Φ the claim is true.

Case 2: There is no unanimous preference of the two alternatives under v^1; then unanimity cannot obtain under v^2 either. As the alternatives are ranked, it must be that $\varphi[v^1] \in \{a, a'\}$ by construction of Φ. Suppose, without loss of generality $\varphi[v^1] = a$. If $a \equiv_i a'$ for all $i \in \mathcal{I}$ then $a = a'$ by richness assumption. So suppose for some individual i, $a' \not\equiv_i a = \varphi[v^1]$. Then, $a\ \Phi(v^1)\ a'$ by construction of Φ. As all individuals rank $\varphi[v^1] = a$ and a' the same way under v^1 and v^2, and φ is auditable then $\varphi[v^2] \neq a'$, by richness. This implies $\varphi[v^2] = a$, as well. Since φ always picks the unique top alternative of the SWF Φ, then $a\ \Phi(v^2)\ a'$. Thus, Φ satisfies IIA. QED

Proof of Proposition 3. To show that auditable φ is non-bossy, let $v \in \mathbf{V}$ and, for an individual i, $v'_i \in \mathbf{V}_i$ be such that

$$\varphi[v] \equiv_i \varphi[v'_i, v_{-i}].$$

Suppose, by way of contradiction, that there is an individual j for whom $\varphi[v'_i, v_{-i}] \not\equiv_j \varphi[v]$. This contradicts auditability between alternative $a = \varphi[v'_i, v_{-i}]$ and $\varphi[v]$ because all individuals rank alternatives a and $\varphi[v]$ in the same way under v and (v'_i, v_{-i}); yet, $\varphi[v'_i, v_{-i}] = a$. Thus, for every $j \in \mathcal{I}$, $\varphi[v'_i, v_{-i}] \equiv_j \varphi[v]$, which in turn implies by richness assumption $\varphi[v'_i, v_{-i}] = \varphi[v]$.

To show that non-bossiness does not in general imply auditability even for Pareto-efficient mechanisms, consider the universal public choice environment. In this domain, every mechanism is non-bossy by definition. Suppose $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$ for some $k \geqslant 3$. Consider the *plurality voting mechanism* φ which selects as the outcome the alternative that is ranked as the top choice by most individuals (and if there are multiple such alternatives then chooses the one with the smallest index[33]). This mechanism is clearly Pareto efficient but not auditable. To see the last point consider two utility profiles v, v' such that their induced profiles are denoted by $o(.)$ function below, with

[33] Formally, let $v_a(\succcurlyeq) = |\{i \in \mathcal{I} : a \in \max_{\succcurlyeq_i} \mathcal{A}\}|$ be the number of *votes* an alternative a gets under a preference profile $\succcurlyeq$; then $\varphi[\succcurlyeq] \in \arg\max_{a \in \mathcal{A}} v_a(\succcurlyeq)$ with the property that $\varphi[\succcurlyeq]$ has the smallest index among all alternatives in the set $\arg\max_{a \in \mathcal{A}} v_a(\succcurlyeq)$.

three individuals, . $\mathcal{I} = \{1, 2, 3\}$ and three alternatives $\mathcal{A} = \{a_1, a_2, a_3\}$:

$$o(v) = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline a_1 & a_2 & a_3 \\ \vdots & a_3 & \dots \\ & a_1 & \end{array} \quad o(v') = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline a_1 & a_3 & a_3 \\ \vdots & \vdots & \vdots \\ & & \end{array},$$

We have $\varphi[v] = a_1$ as all alternatives receive one vote and a_1 has the lowest index, while $\varphi[v'] = a_3$ with the highest votes. On the other hand, v and v' rank relatively a_1 and a_3 the same (individual 1 prefers a_1 while 2 and 3 prefer a_3) yet $\varphi[v'] = a_3 \not\equiv_i \varphi[v]$ for any $i \in \mathcal{I}$. Thus, φ is not auditable.

To show that a group strategy-proof mechanism φ is auditable, consider a utility profile v and any alternative a such that, $a \neq \varphi[v]$. Bu richness, $\varphi[v] \not\equiv_j a$ for some individual j. Let v' be a utility profile such that comparisons of alternatives a and $\varphi[v]$ are the same under v_i and v'_i for every $i \in \mathcal{I}$, i.e., $v_i(\varphi[v]) \geqslant v_i(a) \iff v'_i(\varphi[v]) \geqslant v'_i(a)$. We show that, by richness, $\varphi[v'] \neq a$ (contrapositive of the condition in the auditability definition). Suppose not, i.e., $\varphi[v'] = a \neq \varphi[v]$. Since all individuals rank a and $\varphi[v]$ the same way in both v and v' then a utility profile v^* that for each agent i ranks a and $\varphi[v]$ as the top two choices of hers in the same order as in v_i and $v_i{}'$ is a φ-monotonic transformation of both v and v'. Since φ is monotonic by Proposition 6, we have $\varphi[v^*] = \varphi[v]$ and $\varphi[v^*]= \varphi[v'] = a$, a contradiction to $a \neq \varphi[v]$. Thus $\varphi[v'] \neq a$ and as a result φ is auditable.

To show that an auditable and Pareto efficient mechanism φ does not need to be auditable, consider a problem with 3 agents and 2 alternatives. The mechanism we propose chooses the unanimous top choice if such a unanimous choice exists, and otherwise chooses the second choice alternative of agent 1. This mechanism is by definition Pareto efficient as there are only two alternatives and whenever there is no unanimous choice, there is an agent who likes agent 1's second choice best. This mechanism is auditable as there is a unique preference profile that keeps the order of the two candidates the same.

E.2 Proofs of Results in Section 4.2

Proof of Lemma 1. Consider allocating three or more houses to two individuals. Let φ be a TTC mechanism in which individual 1 owns house A and individual 2 owns houses B and C. To see that there is no complete SWF such that φ is efficient, consider the preference profile

$$\succcurlyeq = \begin{array}{|c|c|} 1 & 2 \\ \hline B & A \\ A & B \\ C & C \\ \vdots & \vdots \end{array}.$$

and the following four auxiliary preference profiles

$$\succcurlyeq^1 = \begin{array}{|c|c|} 1 & 2 \\ \hline B & C \\ A & \vdots \\ \vdots & \end{array}, \quad \succcurlyeq^2 = \begin{array}{|c|c|} 1 & 2 \\ \hline B & B \\ C & C \\ \vdots & \vdots \end{array}, \quad \succcurlyeq^3 = \begin{array}{|c|c|} 1 & 2 \\ \hline C & A \\ \vdots & B \\ & \vdots \end{array}, \quad \succcurlyeq^4 = \begin{array}{|c|c|} 1 & 2 \\ \hline A & A \\ C & C \\ \vdots & \vdots \end{array}.$$

Denote

$$\begin{aligned} a^1 &= \varphi[\succcurlyeq^1] = \{(1,B),(2,C)\}, \\ a^2 &= \varphi[\succcurlyeq^2] = \{(1,C),(2,B)\}, \\ a^3 &= \varphi[\succcurlyeq^3] = \{(1,C),(2,A)\}, \\ a^4 &= \varphi[\succcurlyeq^4] = \{(1,A),(2,C)\}. \end{aligned}$$

Now, if there is a complete SWF Φ such that φ is Arrovian efficient, then $\Phi\left(\succcurlyeq^1\right)$ ranks a^1 above a^4, and by IIA, this implies that $\Phi\left(\succcurlyeq\right)$ ranks a^1 above a^4. Similarly, $\Phi\left(\succcurlyeq^2\right)$ ranks a^2 above a^1, and by IIA, this implies that $\Phi\left(\succcurlyeq\right)$ ranks a^2 above a^1. Further, and again similarly, $\Phi\left(\succcurlyeq^3\right)$ ranks a^3 above a^2, and by IIA, this implies that $\Phi\left(\succcurlyeq\right)$ ranks a^3 above a^2. Finally, $\Phi\left(\succcurlyeq^4\right)$ ranks a^4 above a^3, and by IIA, this implies that $\Phi\left(\succcurlyeq\right)$ ranks a^4 above a^3. But then $\Phi\left(\succcurlyeq\right)$ fails transitivity, showing that there does not exist a complete SWF with respect to which φ is efficient. QED

Proof of Theorem 5. If $|\mathcal{I}| = 1$, the theorem is trivially true. Suppose $|\mathcal{I}| \geqslant 2$.

($\Longrightarrow$) Consider a mechanism φ that is strategy-proof and efficient with respect to a complete Arrovian welfare function. By Proposition 4 and Theorem 4, φ is a TC mechanism $\psi^{\kappa,\beta}$.

Fix an arbitrary preference profile $\succcurlyeq \in \mathbf{R}$. We claim that at any round r of the algorithm $\psi^{\kappa,\beta}$, there is exactly one individual who controls all houses. We prove it in two steps. First, let us show that there cannot be two (or more) individuals who each own a house. By way of contradiction, suppose that some individual 1 controls house A and some other individual 2 controls house B in round r.

Suppose s is the submatching created by the TC algorithm for $\psi^{\kappa,\beta}$ before round r at $\succcurlyeq$. Fix house $C \in \{A, B\}$ as an unmatched house at s. Consider four auxiliary preference profiles $\succcurlyeq^\ell$ that all share the following properties: (i) each individual matched under s ranks houses under $\succcurlyeq^\ell$, $\ell = 1, ..., 4$, in the same way they rank them under $\succcurlyeq$, (ii) each individual i unmatched at s and different from individuals 1 and 2 ranks a unique s-unmatched house $h_i \notin \{A, B, C\} \cup H_s$ as her first choice (such a unique house exists as $|H| > |\mathcal{I}|$), and (iii) individuals 1 and 2 each rank all houses other than A, B, C lower than A, B, C. In particular, the four profiles differ only in how individuals 1 and 2 rank houses A, B, C: the ranking of A, B, C is the same as in the four preference profiles from the proof of Lemma 1 above. Notice that

$$\psi^{\kappa,\beta}[\succcurlyeq^\ell] = s \cup a^\ell \cup \{(i, h_i)\}_{i \in \overline{\mathcal{I}_s} - \{1,2\}},$$

where a^ℓs are defined as in the proof of Lemma 1 above. Furthermore, the same argument we used in the proof of the lemma shows that there can be no SWF that ranks all four a^ℓs, is transitive, and satisfies IIA. Hence, there is no complete SWF that makes $\psi^{\kappa,\beta}$ efficient, a contradiction that implies that there cannot be two individuals who own houses in a round of the algorithm.

As $\psi^{\kappa,\beta}$ never allows two owners in a round of the algorithm, by Theorem 4 and Remark 1, without loss of generality we can assume that there are no brokers in any round, either. Hence, in each round of the algorithm there is a single individual who controls (and owns) all houses. That means that $\psi^{\kappa,\beta}$ is a sequential dictatorship.

($\Longleftarrow$) Consider a sequential dictatorship ψ^κ. We construct a complete SWF Φ such that ψ^κ is efficient with respect to Φ. Under Φ any two matchings are ranked according to the preference relation of the first-round dictator; if she is indifferent , then the matchings are ranked according to the preference relation of the second-round dictator, etc. Formally, for any $\succcurlyeq \in \mathbf{R}$ and any two distinct $a, b \in A$, let $a\,\Phi(\succcurlyeq)\,\beta$ if, and only if, there exists $k \in \{1, ..., |\mathcal{I}|\}$ such that $a(i_1) = b(i_1)$, ... and $a(i_{k-1}) = b(i_{k-1})$, and individual i_k prefers $a(i_k)$ over $b(i_k)$, where individuals $i_1, ..., i_k$ are defined recursively: $i_1 = \kappa(\varnothing)$, and in general $i_\ell = \kappa(\{(i_1, a(i_1)), ..., (i_{\ell-1}, a(i_{\ell-1}))\})$ for $\ell = 1, ..., k$. It is straightforward to verify that Φ is a complete Arrovian SWF and that ψ^κ is efficient with respect to Φ. QED

Lemma 2 *Suppose that $|H| = |\mathcal{I}| \geqslant 3$ and a TC mechanism is Arrovian efficient with respect to a complete SWF. Then in this mechanism one individual cannot control a house while two others each own a house.*

Proof of Lemma 2. Consider a TC mechanism φ in which individual 1 owns house A, individual 2 owns house B, and individual 3 controls house C. We will show that there is no complete SWF such that φ is Arrovian efficient. Consider the preference profile

$$\succcurlyeq = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & C & A \\ C & A & B \\ A & B & C \\ \vdots & \vdots & \vdots \end{array}.$$

and the following three additional preference profiles

$$\succcurlyeq^1 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & C & B \\ C & \vdots & \vdots \\ A & & \\ \vdots & & \end{array}, \quad \succcurlyeq^2 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline C & C & A \\ \vdots & A & \vdots \\ & B & \\ & \vdots & \end{array}, \quad \succcurlyeq^3 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & A & A \\ \vdots & \vdots & B \\ & & C \\ & & \vdots \end{array}.$$

Regardless of whether individual 3 owns or brokers house C, we have

$$\begin{aligned}
a^1 &= \varphi[\succcurlyeq^1] = \{(1,A),(2,C),(3,B)\};\\
a^2 &= \varphi[\succcurlyeq^2] = \{(1,C),(2,B),(3,A)\};\\
a^3 &= \varphi[\succcurlyeq^3] = \{(1,B),(2,A),(3,C)\}.
\end{aligned}$$

If there is a complete SWF Φ such that φ is Arrovian efficient, then $\Phi\left(\succcurlyeq^1\right)$ ranks a^1 above a^3, and by IIA, this implies that $\Phi\left(\succcurlyeq\right)$ ranks a^1 above a^3. Similarly, $\Phi\left(\succcurlyeq^2\right)$ ranks a^2 above a^1, and by IIA, this implies that $\Phi\left(\succcurlyeq\right)$ ranks a^2 above a^1. Further, and again similarly, $\Phi\left(\succcurlyeq^3\right)$ ranks a^3 above a^2, and by IIA, this implies that $\Phi\left(\succcurlyeq\right)$ ranks a^3 above a^2. Then $\Phi\left(\succcurlyeq\right)$ fails transitivity, showing that there does not exist a complete SWF with respect to which φ is efficient. QED

Lemma 3 *Suppose that $|H| = |\mathcal{I}| \geqslant 3$ and a TC mechanism is Arrovian efficient with respect to a complete SWF. Then, in any round of the TC algorithm, there is at most one broker.*

Proof of Lemma 3. By way of contradiction, suppose that in some round of the TC mechanism there are more than one broker and let φ be the continuation TC mechanism from this round onwards. Without loss of generality, in φ individual 1 brokers house A, individual 2 brokers house B, and individual 3 brokers house C. We will show that there is no complete SWF such that φ is Arrovian efficient. Consider the following preference profiles

$$\succcurlyeq = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & B & C \\ A & A & B \\ C & C & A \\ \vdots & \vdots & \vdots \end{array}.$$

and

$$\succcurlyeq^1 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline A & B & C \\ C & A & B \\ \vdots & \vdots & \vdots \end{array},\quad \succcurlyeq^2 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & B & C \\ A & C & A \\ \vdots & \vdots & \vdots \end{array},\quad \succcurlyeq^3 = \begin{array}{|c|c|c|} 1 & 2 & 3 \\ \hline B & A & B \\ C & C & A \\ \vdots & \vdots & \vdots \end{array}.$$

Denote

$$\begin{aligned}
a^1 &= \varphi[\succcurlyeq^1] = \{(1,A),(2,B),(3,C)\};\\
a^2 &= \varphi[\succcurlyeq^2] = \{(1,B),(2,C),(3,A)\};\\
a^3 &= \varphi[\succcurlyeq^3] = \{(1,C),(2,A),(3,B)\}.
\end{aligned}$$

If there is a complete SWF Φ such that φ is Arrovian efficient, then $\Phi\left(\succcurlyeq^1\right)$ ranks a^1 above a^3, and by IIA, this implies that $\Phi(\succcurlyeq)$ ranks a^1 above a^3. Similarly, $\Phi\left(\succcurlyeq^2\right)$ ranks a^2 above a^1, and by IIA, this implies that $\Phi(\succcurlyeq)$ ranks a^2 above a^1. Further, again similarly, $\Phi\left(\succcurlyeq^3\right)$ ranks a^3 above a^2, and by IIA, this implies that $\Phi(\succcurlyeq)$ ranks a^3 above a^2. Then $\Phi(\succcurlyeq)$ fails transitivity, showing that there does not exist a complete SWF with respect to which φ is efficient. QED

Proof of Theorem 6. If $|H| > |\mathcal{I}|$, it follows from Theorem 5 and if $|H| = |\mathcal{I}| = 1$, the theorem is trivially true. Hence, suppose $|H| = |\mathcal{I}| > 1$.

($\Longrightarrow$) Consider a mechanism φ that is strategy-proof and efficient with respect to a complete Arrovian welfare function. By Proposition 4 and Theorem 4, φ is a TC mechanism $\psi^{\kappa,\beta}$.

Fix $\succcurlyeq \in \mathbf{R}$. We claim that at any round r of the algorithm for $\psi^{\kappa,\beta}$, there is exactly one individual who controls all houses whenever $|\overline{\mathcal{I}_s}| > 2$. We prove it in three steps (in accordance with Lemmas 1-3). Let s be the submatching created by the algorithm $\psi^{\kappa,\beta}$ before round r for $\succcurlyeq$.

- First, we show that an individual cannot own two houses while another individual owns a third house: By way of contradiction, suppose that some individual 1 owns house A and individual 2 owns houses B and C in round r. Then there exists an individual 3 who does not control any house at round r as $|H| = |\mathcal{I}|$. Consider four auxiliary preference profiles $\succcurlyeq^\ell$ that all share the following properties: (i) each individual matched under s ranks houses under $\succcurlyeq^\ell$, $\ell = 1, ..., 4$, in the same way they rank them under $\succcurlyeq$, (ii) each individual i unmatched at s and different from individuals $1, 2, 3$ ranks a unique s-unmatched house $h_i \notin \{A, B, C\} \cup H_s$ as her first choice (such a unique house exists as $|H| = |\mathcal{I}|$), (iii) individuals 1 and 2 each rank all houses other than A, B, C lower than A, B, C, and (iv) individual 3's preference relation is the same as $\succcurlyeq_3$ under all four profiles. In particular, the four profiles differ only in how individuals 1 and 2 rank houses A, B, C: the ranking of A, B, C is the same as in the four preference profiles of the proof of Lemma 1 above. Notice that

$$\psi^{\kappa,\beta}[\succcurlyeq^\ell] = s \cup a^\ell \cup \{(i, h_i)\}_{i \in \overline{\mathcal{I}_s} - \{1,2,3\}},$$

where a^ℓs are defined as in the proof of Lemma 1 above. Furthermore, the same argument we used in the proof of Lemma 1 shows that there can be no SWF that ranks all four a^ℓs, is transitive, and satisfies IIA. Hence, there is no complete SWF that makes $\psi^{\kappa,\beta}$ efficient, a contradiction.

- Next, we show that one individual cannot control a house while at least two others each own a house in round r: Suppose, to the contrary, individual 1 owns house A, individual 2 owns house B, and individual 3 controls house C in round r. Consider three auxiliary preference profiles $\succcurlyeq^\ell$ that all share the following properties: (i) each individual matched under s ranks houses under $\succcurlyeq^\ell$, $\ell = 1, 2, 3$, in the same way they rank them under $\succcurlyeq$, (ii) each individual i unmatched at s and different from individuals $1, 2, 3$ ranks a unique s-unmatched house $h_i \notin \{A, B, C\} \cup H_s$ as her first choice (such a unique house exists as $|H| = |\mathcal{I}|$), and (iii) individuals 1,2, 3 each rank all houses other than A, B, C lower than A, B, C, and the ranking of A, B, C is the same as in the three preference profiles of the proof of Lemma 2 above. Observe that

$$\psi^{\kappa,\beta}[\succcurlyeq^{\ell}] = s \cup a^{\ell} \cup \{(i, h_i)\}_{i \in \overline{\mathcal{I}_s} - \{1,2,3\}},$$

where a^{ℓ}s are defined as in the proof of Lemma 2 above. Furthermore, the same argument we used in the proof of Lemma 2 shows that there can be no SWF that ranks all three a^{ℓ}s, is transitive, and satisfies IIA. Hence, there is no complete SWF that makes $\psi^{\kappa,\beta}$ efficient, a contradiction.

- Finally, using a variant of Lemma 3, we show that there cannot be multiple brokers at round r (as multiple brokers can only occur with 3 individuals and 3 houses, where each individual brokers a distinct house): Suppose not. Then consider three auxiliary preference profiles $\succcurlyeq^{\ell}$ that all share the following properties: (i) each individual matched under s ranks houses under $\succcurlyeq^{\ell}$, $\ell = 1, 2, 3$, in the same way they rank them under $\succcurlyeq$, (ii) individuals 1,2, 3, who are the only remaining unmatched individuals, each rank all houses other than A, B, C lower than A, B, C, and (iii) the ranking of A, B, C is the same as in the three preference profiles of the proof of Lemma 3 above. Notice that

$$\psi^{\kappa,\beta}[\succcurlyeq^{\ell}] = s \cup a^{\ell},$$

where a^{ℓ}s are defined as in the proof of Lemma 3 above. Furthermore, the same argument we used in the proof of Lemma 3 shows that there can be no SWF that ranks all three a^{ℓ}s, is transitive, and satisfies IIA. Hence, there is no complete SWF that makes $\psi^{\kappa,\beta}$ efficient, a contradiction.

Thus, a single individual owns all houses at round r when s is fixed for $|\overline{\mathcal{I}_s}| > 2$ (by Theorem 4 and Remark 1).

This means that $\psi^{\kappa,\beta}$ is an almost sequential dictatorship, as all TC mechanisms restricted to only two individuals are almost sequential dictatorships.

($\Longleftarrow$) Consider an almost sequential dictatorship ψ^{κ}. If ψ^{κ} is a sequential dictatorship, then the proof of Theorem 5 works. So suppose it is not a sequential dictatorship. Hence, $|H| = |\mathcal{I}|$. We construct a complete SWF Φ such that ψ^{κ} is efficient with respect to Φ. Under Φ any two matchings are ranked according to the preference relation of the first-round dictator; if she is indifferent , then the matchings are ranked according to the preference relation of the second-round dictator, etc., until only two individuals remain unmatched. At this round let 1 and 2 be the two individuals and A and B be the two houses remaining unmatched. Observe that there are only two matchings, a and b, in which all individuals' assignments are the same but the last two: in one 1 gets A and 2 gets B, and in the other vice versa. Then one of these two matchings is equal to $\psi^{\kappa}[\succcurlyeq']$, where $\succcurlyeq'$ ranks the assignment of any individual other than 1 and 2 in a (or equivalently b) as her first choice, and for 1 and 2, the new preferences are the same as the original ones under $\succcurlyeq$. We rank $\psi^{\kappa}[\succcurlyeq'] \in \{a, b\}$ before the other one under $\Phi(\succcurlyeq)$.

Formally, for every $a \in \mathcal{A}$, let sequential dictators $i_1, \ldots, i_{|\mathcal{I}|-2}$ be defined as $i_1 = \kappa_h(\emptyset)$ for every $h \in H$, and in general, $i_\ell = \kappa_h(\{(i_1, a(i_1)), ..., (i_{\ell-1}, a(i_{\ell-1}))\})$ for every $h \in H - \{a(i_1), ...a(i_{\ell-1})\}$ and $\ell = 1, ..., k$; then for every $b \in \mathcal{A} - \{a\}$, we say $a\, \Phi(\succcurlyeq)\, b$ if one of the following two conditions holds:

1. there exists $k \in \{1, ..., |\mathcal{I}| - 2\}$ such that $a(i_1) = b(i_1)$, ..., $a(i_{k-1}) = b(i_{k-1})$, and $a(i_k) \succ_{i_k} b(i_k)$; or

2. for every $\ell \in \{1, ..., |\mathcal{I}| - 2\}$, $a(i_\ell) = b(i_\ell)$, and for $\succcurlyeq' \in \mathbf{R}$ where each i_ℓ ranks $a(i_\ell)$ first while the remaining two individuals have the same preferences as in $\succcurlyeq$, we have $\psi^\kappa[\succcurlyeq'] = a$.

By construction, Φ is complete, antisymmetric, and transitive. Moreover, it satisfies the Pareto postulate. To see that it also satisfies IIA, consider two distinct matchings, $a, b \in \mathcal{A}$, and $\succcurlyeq \in \mathbf{R}$ such that $a\ \Phi(\succcurlyeq)\ b$. Also consider another profile $\hat{\succcurlyeq} \in \mathbf{R}$ such that each individual i's preference over the two matching assignments is the same in $\hat{\succcurlyeq}_i$ as in $\succcurlyeq_i$. If $a\ \Phi(\succcurlyeq)\ b$ because of condition 1 above, then condition 1 continues to hold for $\hat{\succcurlyeq}$ and thus $a\ \Phi(\hat{\succcurlyeq})\ b$. On the other hand, if $a\ \Phi(\succcurlyeq)\ b$ because of condition 2 above, then a and b only differ in how the last two individuals are assigned the remaining two houses. Hence, the profile constructed to check condition 2 for $a\ \Phi(\hat{\succcurlyeq})\ b$, which we refer to as $\hat{\succcurlyeq}'$, would lead to $\psi^\kappa[\hat{\succcurlyeq}'] = a$ because:

1. the first $|\mathcal{I}| - 2$ dictators would still get their a assignments in the first $|\mathcal{I} \vdash 2$ rounds of the TC algorithm for $\psi^\kappa[\hat{\succcurlyeq}']$, and

2. thus, the assignment of remaining two individuals under $\psi^\kappa[\hat{\succcurlyeq}']$ would be identical with that under a as the relative ranking of their assignments under a and b are identical both in $\succcurlyeq$ and $\hat{\succcurlyeq}$, and the ranking of the other houses do not matter for finding the outcome of the almost serial dictatorship.

Thus, $a\ \Phi(\hat{\succcurlyeq})\ b$, showing Φ satisfies IIA. QED