Strategy-proof, Efficient, and Fair Allocation: Beyond Random Priority[*]

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Abstract

Random Priority is a popular mechanism used to allocate a set of objects to a set of agents without the use of monetary transfers. Random Priority is appealing because it satisfies desirable efficiency (Pareto efficiency), fairness (equal treatment of equals), and incentive (strategy-proofness) properties. Is it the only mechanism with these properties? This has been a long-standing open question and in this note we answer it by constructing other mechanisms (*Correlated Random Priority* mechanisms) satisfying these desirable properties.

1 Introduction

Consider the problem of allocating n indivisible objects to n agents without the use of monetary transfers. Examples of such problems include assigning school seats to K12 students, dormitory rooms to college students, tasks to workers, offices to professors, or time slots on a common machine. A classic and oft-used solution to this problem is the *Random Priority (RP)* mechanism: an ordering of the agents is drawn uniformly at random, and agents are called, one-by-one, to select their favorite object from those that were not selected by earlier agents.[1] The popularity of RP largely derives from its desirable efficiency, fairness, and incentive (or simplicity) properties:

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[1]Random Priority also goes by the name Random Serial Dictatorship, see e.g., Abdulkadiroğlu and Sönmez (1998).

- *Pareto efficiency:* for any preferences of the agents, the final allocation of RP is Pareto efficient.
- *Equal treatment of equals:* if two agents have the same preferences, RP assigns them the same distribution over outcomes; in other words, RP satisfies.
- *Strategy-proofness:* irrespective of the reports of others, it is always in an agent's best interest to report their preferences over the objects truthfully.

While it is straightforward to show that RP satisfies the above properties, an open question is whether any *other* mechanism also satisfies these properties, or whether RP is the unique mechanism to do so. Many authors attempted to resolve this conjecture.[2] We provide a resolution by constructing an alternative strategy-proof, Pareto efficient, and equal-treatment mechanism that differs from RP.

The construction gives us a large class of mechanisms that we call *Correlated Random Priority* mechanisms. All mechanisms in this class satisfy the above key axioms but the class offers more flexibility in correlating outcomes of individual agents.

While our construction resolves the above strong version of the conjecture, its weaker versions might still be true. Weaker versions of the conjecture weaken one of the assumptions or relax the uniqueness claim. RP fails the stronger efficiency properties of ex-ante efficiency (Zhou, 1990) and ordinal efficiency (Bogomolnaia and Moulin, 2001).[3] RP satisfies stronger fairness property of symmetry (Pycia and Troyan, 2020) as well as stronger incentive and simplicity properties such as obvious strategy-proofness (Li, 2017), one-step simplicity, and strong obvious strategy-proofness (Pycia and Troyan, 2020). Uniqueness claim can be relaxed to the uniqueness of each agent's marginal outcome distribution (as in the positive results of Liu and Pycia 2011).

The sole weak version of the conjecture so far proven is Pycia and Troyan's (2020) result that RP is the unique mechanism that is Pareto efficient, symmetric, and obviously strategy-proof. An earlier step towards proving the conjecture was made by Abdulkadiroğlu and Sönmez (1998) and Knuth (1996) who showed that RP is equivalent to another mechanism called the core from random endowments, which works by

[2]For main analyses that led to results weaker than the above conjecture, see the literature discussion. Pycia and Ünver (2015) discuss methodological tools developed in a failed attempt to prove the conjecture.

[3]RP satisfies ordinal efficiency asymptotically as established in Che and Kojima (2010); cf. Liu and Pycia (2011) for general asymptotic equivalence of Pareto and ordinal efficiency. RP fails ex-ante efficiency even asymptotically, see Abdulkadiroğlu et al. (2009), Featherstone and Niederle (2008), and Miralles (2008).

first randomly assigning the objects to the agents and then allowing the agents to trade according to the top trading cycles (TTC) algorithm of Shapley and Scarf (1974). This equivalence result has been extended, e.g., by Pathak and Sethuraman (2011), Carroll (2014), and Pycia (2016).

Earlier work also established an asymptotic version of the conjecture: Liu and Pycia (2011) showed that asymptotically, in large markets, all ordinally efficient, equal treatment, strategy-proof mechanisms with small agents have the same marginal distributions as RP. The characterizations based on ordinal efficiency cannot however be extended to finite markets because Bogomolnaia and Moulin (2001) showed that there is no ordinally efficient, strategy-proof, and ETE mechanism when $n \geq 4$.[4]

2 Model

We consider the problem of allocating n indivisible objects to n agents. We let X denote the set of objects and I denote the set of agents. Each agent $i \in I$ has a strict **preference relation** P_i over X, where we write xP_iy to denote that x is strictly preferred to y, and xR_iy if either xP_iy or $x = y$. We use $\mathcal{P}$ to denote the set of all strict preference relations over X. We use $P_I = (P_i)_{i \in I}$ to denote a profile of preferences, one for each agent, and $\mathcal{P}^n$ to denote the set of all preference profiles. A deterministic **allocation** $a : I \to X$ is a one-to-one function, where $a(i)$ is the object allocated to agent i. We let $\mathcal{A}$ denote the set of allocations. A **random allocation** $\mu : \mathcal{A} \to [0, 1]$ is a probability distribution over $\mathcal{A}$, where $\sum_{a \in \mathcal{A}} \mu(a) = 1$. We let $\mathcal{M}$ denote the set of random allocations.

A **mechanism** $\psi : \mathcal{P}^n \to \mathcal{M}$ is a mapping from preference profiles of the agents to random allocations. Given a mechanism ψ, we write $\psi(P_I)(a)$ to denote the probability that allocation a is implemented when the preferences are P_I. Let $\pi_i^k(\psi(P_I)) = \sum_{a \in \mathcal{A}} \psi(P_I)(a)\mathbb{1}\{a(i) = x_k\}$ be the probability that i receives object x_k at the random allocation $\psi(P_I)$. Finally, we write $\psi_i(P_I) = (\pi_i^k(\psi(P_I)))_{k=1}^n$ to be i's lottery over objects under mechanism ψ at preference profile P_I.

We are interested in the following canonical efficiency, fairness, and incentive properties:

- *Efficiency*: A deterministic allocation a is **Pareto efficient** if there is no other

[4] Cf. also Zhou (1990) who shows that there is no strategyproof, ETE, and ex-ante efficient mechanism for $n \geq 3$. Erdil 2014 shows that the equality of the number of agents and objects plays a crucial role in the conjecture.

allocation a' such that $a(i) R_i a'(i)$ for all $i \in I$ and $a(i) P_i a'(i)$ for some $i \in I$. A mechanism ψ is **Pareto efficient** if, for all $P_I \in \mathcal{P}^n$, every deterministic allocation in the support of $\psi(P_I)$ is Pareto efficient.

- *Fairness:* A mechanism ψ satisfies **equal treatment of equals (ETE)** if for all $P_I \in \mathcal{P}^n$, $P_i = P_j$ implies $\psi_i(P_I) = \psi_j(P_I)$.

- *Incentives:* A mechanism ψ is **strategyproof** if $\psi_i(P_i, P_{-i})$ first-order stochastically dominates $\psi_i(P'_i, P_{-i})$ for all $P_i, P'_i \in \mathcal{P}$ and all $P_{-i} \in \mathcal{P}^{n-1}$, where first-order stochastic dominance is defined with respect to i's true preferences P_i.

We say that two mechanisms ψ and ϕ are **equivalent** if $\psi(P_I) = \phi(P_I)$ for all P_I.

The **Random Priority (RP)** mechanism works as follows. An agent ordering is drawn uniformly at random from the set of all permutations of I. Agents are then assigned objects in this order, with each agent receiving her most preferred object (according to her reported preferences) among the set of objects that have not been assigned to earlier agents. For any preference profile P_I, we define $\psi^{RP}(P_I)$ as the lottery over $\mathcal{A}$ induced by this procedure. It is well-known that ψ^{RP} is strategyproof, Pareto efficient, and satisfies equal treatment of equals.

3 Result: Beyond Random Priority

We are now ready to prove

Theorem 1. *There exists a Pareto efficient, strategyproof, and ETE mechanism ψ that is not equivalent to Random Priority.*

Proof. We prove the theorem using a counterexample with 4 agents, $I = \{1, 2, 3, 4\}$, and 4 objects, $X = \{w, x, y, z\}$. For shorthand, we write $P_i : w, x, y, z$ to denote that i strictly prefers w to x to y to z.

Fix a profile of preferences P_I, and consider the following algorithm.

- Draw an ordering of the agents uniformly from the set of all permutations of I. Denote this ordering as $\sigma : \sigma_1, \sigma_2, \sigma_3, \sigma_4$.

- If $P_1 = P_2 = w, x, y, z$, $\sigma_1 = 1$, and $\sigma_2 = 2$, then assign agent 1 to w, agent 2 to x, agent 3 to her top choice among $\{y, z\}$ (according to P_3), and agent 4 to the remaining unassigned object.

- If $P_1 = P_2 = w, x, y, z$, $\sigma_1 = 2$, and $\sigma_2 = 1$, then assign agent 2 to w, agent 1 to x, agent 4 to her top choice among $\{y, z\}$ (according to P_4), and agent 3 to the remaining unassigned object.

- In all other cases, assign agents in the order $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ to their favorite object among those that were not selected by earlier agents (the entire set X for agent σ_1).

Define ψ as the mechanism that results from applying the above algorithm to any preference profile P_I. Note that this is very similar to standard Random Priority, except in two special cases described in the second and third bullet points. In these specific instances, the third agent to select is not chosen randomly from the remaining agents.

It is trivial to see that this mechanism is Pareto efficient, as the algorithm always results in a Pareto efficient deterministic allocation. It is also easy to see that the mechanism is strategy-proof: agents cannot affect their place in the selection order, and at their turn, it is optimal to have reported their true preferences.

For equal treatment of equals, note first that it is well-known that RP satisfies ETE. Further, on any preference profile P_I in which either $P_1 \neq w, x, y, z$ or $P_2 \neq w, x, y, z$, $\psi(P_I)$ produces the same lottery over deterministic allocations as RP (and therefore immediately satisfies ETE on all such profiles). Thus, consider any P_I where $P_1 = P_2 = w, x, y, z$. For all σ such that $\{\sigma_1, \sigma_2\} \neq \{i_1, i_2\}$, ψ once again leads to the same deterministic allocation as the corresponding case under RP.

Thus, there are 4 cases left, $\sigma : 1, 2, 3, 4$, $\sigma' : 1, 2, 4, 3$, $\sigma'' : 2, 1, 3, 4$, and $\sigma''' : 2, 1, 4, 3$. It is obvious that 1 and 2 receive the same allocations under each of these as under RP, and so $\psi_1(P_I)$ and $\psi_2(P_I)$ are the same as 1 and 2's lotteries under RP. Finally, consider agents 3 and 4, and note that since 1 and 2 will take w and x, only the relative rankings of y and z matter. If P_3 and P_4 rank y and z differently, then they each receive their favorite from the set $\{y, z\}$, which once again is the same as under RP. So, consider the case that both P_3 and P_4 prefer y to z. Note that under both RP and ψ, 3 receives y and 4 receives z for exactly 2 of $\{\sigma, \sigma', \sigma'', \sigma'''\}$, and the allocation is reversed for the other 2 selections: under RP, 3 receives y and 4 receives z under σ and σ'', and vice-versa under σ' and σ''', while under ψ, 3 receives y and 4 receives z under σ and σ', while 4 receives y and 3 receives z under σ'' and σ'''. The remaining case where both 3 and 4 prefer z to y is analogous, and so, summing up, we conclude that both $\psi_3(P_I)$ and $\psi_4(P_I)$ are equivalent to the respective lotteries under RP. Therefore,

ψ also satisfies ETE.

Finally, we argue that ψ is not equivalent to RP. To see this, consider the preference profile $P_i = w, x, y, z$ for all $i \in I$, and the following allocation:

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ w & x & z & y \end{pmatrix}.$$

Note that $\psi(P_I)(a) = 0$, while $\psi^{RP}(P_I)(a) > 0$. Therefore, the two mechanisms are not equivalent. ■

The above proof makes it clear that a large class of mechanisms—that we call *Correlated Random Priority* (*CRP*)—satisfy the three axioms from Theorem 1. In defining CRP mechanisms we use the auxiliary notions of the top of an ordering, continuation ordering, and of coupling. We say that a set of agents J is at the top of an ordering if the ordering ranks agents from J at the top, that is $\left\{\sigma_1, ..., \sigma_{|J|}\right\} = J$. By continuation ordering we mean the restriction $\sigma_{\{k+1,...,|I|\}}$ of an ordering σ to its last $|I| - k$ positions. We say that a set of $k \geq 2$ orderings $S = \{\sigma_1, ..., \sigma_k\}$ of agent is coupled at set of agents J at preference profile P_J of these agents if (i) J is at the top of all orderings in S, and (ii) for any two orderings $\sigma, \sigma' \in S$ the serial dictatorship ψ_σ assigning objects in the order σ and the serial dictatorship $\psi_{\sigma'}$ assigning objects in the order σ' lead to the same objects being assigned to agents from J that is $\psi_\sigma\left(P_J\right)(J) = \psi_{\sigma'}\left(P_J\right)(J)$.[5]

Any CRP mechanism is constructed as follows: (1) we start by assigning each ordering σ and preference profile P_J of agents at the top of σ probability $p\left(\sigma; P_J\right) = \frac{1}{|X|!}$; (2) for any preference profile P_I , set of agents $J \subsetneq I$, and set of orderings S that is coupled at J and P_J, we reshuffle probabilities $p\left(\cdot; P_J\right)$ of orderings in S so that for any continuation ordering $\sigma^*_{\{|J|+1,...,|I|\}}$, the sum of probabilities $\sum_{\left\{\sigma:\sigma_{\{|J|+1,...,|I|\}}=\sigma^*_{\{|J|+1,...,|I|\}}\right\}} p\left(\sigma; P_J\right)$ is unchanged by the reshuffle; otherwise the reshuffle is arbitrary. We repeat step (2) an arbitrary number of times subject to the constraint that after we reshuffled probabilities for some set J and profile P_J, we do not later reshuffle them for any proper subset $J' \subsetneq J$ and $P_{J'}$ such that $P_{J'} = P_J|_{J'}$. The CRP mechanism than draws the first agent uniformly at random and assigns this agent his or her top choice; following a sequence of such assignments to any agents $i_1, ..., i_k$ with preferences $P_{i_1,...,i_k}$ we then draw the next agent so that the probability $i \notin \{i_1, ..., i_k\}$ being drawn is equal to the sum of probabilities $p\left(\sigma; P_{\{i_1,...,i_k\}}\right)$ over all orderings with $\{i_1, ..., i_k\}$ at the top and

[5]We simplify the notation and specify only preferences of agents in J as condition (i) guarantees that the assignments of objects to these agents depend only on their preferences under both ψ_σ and $\psi_{\sigma'}$.

agent i in the $k+1$ position ($\sigma_{k+1} = i$).

We can see that the resulting CRP mechanisms are strategy-proof via two separate arguments. First, the strategy-proofness obtains because no agent i affects the distribution of the sets of agents drawn before i nor the probability that i is drawn following any set of agents J. Second, the strategy-proofness obtains because the reshuflings do not change any agent's marginal distribution of assigned objects; thus the distribution is the same as under Random Priority. The second argument also establishes that any CRP satisfies equal treatment of equals. Finally, the mechanism is Pareto efficient because each of its ex post assignments is obtained via a serial dictatorship.

Note that while the above proof constructs a mechanism ψ that produces a different distribution over allocations of objects to all individuals than ψ^{RP}, from the perspective of an individual agent i, the marginal distributions of i's assignment over individual objects are the same for both mechanisms. Whether all strategy-proof, Pareto efficient, and ETE mechanisms are equivalent to RP under a weaker such notion of marginal equivalence remains an open question.

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