Non-bossiness and First-Price Auctions*

Marek Pycia† Madhav Raghavan‡

June 2021

Abstract

We show that the first-price auction with no reserve price is the essentially unique mechanism that is non-bossy, individually rational, and efficient in equilibrium. The first-price auction with optimal reserve price is the essentially unique mechanism that is non-bossy, individually rational, and revenue maximizing.

1 Introduction

First-price auctions are becoming increasingly popular. For instance, between 2017 and 2019, all the major online ad exchanges—AppNexus, Index Exchange, OpenX, Rubicon Project, PubMatic, and most recently Google AdX—changed their auction mechanisms from a second-price auction to a first-price auction, in order to "provide additional auction transparency to both publishers and advertisers" about how their bids were used and ad allocations were determined.[1] At the root of the change was that in the second-price auction, the winner's payment depends on the bids of others, notably that of the second-highest bidder, which led to suspicions among bidders that the resulting price might be manipulated. The ability of other bidders (including shill bidders who represent the seller) to manipulate the outcome of a mechanism at no cost to themselves is known as *bossiness*.[2]

*First draft: October 2018. First posted draft: November 2020. *JEL Codes:* C72, D44.

†Department of Economics, University of Zurich, Switzerland.

‡Department of Economics, University of Lausanne, Switzerland. Raghavan acknowledges financial support from the Swiss National Science Foundation (100018_162606).

[1] "An update on first-price auctions for Google Ad Manager", 10 May 2019, https://www.blog.google/products/admanager/update-first-price-auctions-google-ad-manager/ (last accessed 02 October 2020), and "Big Changes Coming To Auctions, As Exchanges Roll The Dice On First-Price," 5 September 2017, https://www.adexchanger.com/platforms/big-changes-coming-auctions-exchanges-roll-dice-first-price/ (last accessed 10 November 2020).

[2] The term is due to Satterthwaite and Sonnenschein (1981). In a second-price auction, the second-highest bidder can often influence the outcome by raising her in such a way that she continues to lose the auction and pay nothing, but the raised bid affects the winning price—and thus the outcome—for the highest bidder. For a recent analysis of this influence in the second-price auction, see e.g. Raghavan (2020).

The first-price auction avoids this problem because it is *non-bossy*; it allows no bidders to change the outcome of others without changing their own. In the absence of a reserve price, the first-price auction is also *efficient*, in that it always awards the object to (one of the) highest bidder(s). The first-price auction is *ex post individually rational*, in that in equilibrium no bidder is worse off from participating then from staying out of the auction; the individual rationality obtains because losing bidders pay nothing, and the winning bidder pays the equilibrium bid which is weakly lower than the bidder's value. The first-price auction is also ex post individually rational for the seller, who in equilibrium makes no positive transfers to losing bidders.

This note shows that in the canonical single-object-for-sale private values environment, when utilities for bidders are quasilinear, the first-price auction is the essentially unique auction format whose equilibrium satisfies the above three properties: non-bossiness, efficiency, and individual rationality. Our approach builds on the revenue-equivalence result of Riley and Samuelson (1981) and Myerson (1981) which implies that in any efficient auction, the bidders' interim utility is the same as in the first-price auction. We use a similar approach to show that the first-price auction with optimal reserve price is the essentially unique mechanism that is non-bossy, individually rational, and revenue maximizing.

We contribute to a growing literature on first-price auctions. Most closely related to ours are two recent characterizations of first-price auctions. Akbarpour and Li (2020) characterize the first-price format as the unique static, credible, revenue-maximizing auction format in which only the winner pays.[3] In a non-quasilinear setting, Adachi and Kongo (2013) characterize first-price auctions in terms of efficiency, individual rationality, anonymity in welfare (or envy-freeness) and non-bossiness in welfare. We also contribute to the literature on characterizing efficient, individual rational, and non-bossy mechanisms, cf. Pápai (2000), Pycia and Ünver (2017) and Root and Ahn (2020) who study these mechanisms in settings without transfers.

2 The Model

There is a finite set of risk-neutral *buyers* or *bidders* (the terms are used interchangeably) $N = \{1, 2, \ldots, n\}$ and one seller who owns a single copy of an indivisible good and has a reservation price of zero. Each buyer $i \in N$ independently draws a *valuation* $v_i \in [0, 1]$ for the good according

[3]Credibility captures auctioneer's incentive-compatibility; it is related to self-auditability (Woodward (2020)) and transparency (Hakimov and Raghavan (2020)).

to a Lebesgue-continuous distribution with full support that is common to all bidders.[4] A *valuation profile* is denoted $v \equiv (v_i)_{i \in N}$ and the set of profiles is denoted $V \equiv [0,1]^n$.

An *item allocation* is a vector $x \in \{0,1\}^n$ such that $\sum_{i \in N} x_i \leq 1$. For any $i \in N$, x_i denotes the *assignment* of i in x. The interpretation of $x_i = 1$ is that i is assigned the good; the good is indivisible, and only one agent is assigned the good in any item allocation. A *payment profile* is given by a vector $p \in \mathbb{R}^n$, where p_i denotes the *payment* of buyer $i \in N$. An *outcome* is given by a pair (x, p), where x is an item allocation and p is a payment profile. Let the set of outcomes be denoted $\mathcal{O}$. We assume utilities are quasilinear: for any outcome (x, p), any agent $i \in N$, and any valuation v_i, $u_i(x, p) = x_i v_i - p_i$.

Let M_i denote a set of *messages* for bidder i, with generic element m_i. Let $M \equiv \times_{i \in N} M_i$. A collection of messages $m = (m_i)_{i \in N} \in M$ is called a *message profile*. Let $G : M \to \Delta^{\mathcal{O}}$ be a function that, for each $m \in M$, specifies a probability distribution $G(m)$ over the set of outcomes $\mathcal{O}$. We denote the probability of outcome $(x, p) \in \mathcal{O}$ under $G(m)$ as $P_m(x, p)$. The *support* of $G(m) = \{(x, p) \in \mathcal{O} \mid P_m(x, p) > 0\}$. To avoid trivialities, we require that for any $(x, p), (x', p')$ in the support of $G(m)$ and any $i \in N$, we have $x = x' \implies p = p'$; that is the assignment vector determines the payment vector.

The pair (M, G) represents an *auction game*. A *strategy* for $i \in N$ is a function $\theta_i : V \to M_i$ that specifies a message $\theta_i(v) \in M_i$ for each $v \in V$. Let $\theta \equiv (\theta_i)_{i \in N}$ be a *strategy profile*. A *mechanism* (or incentive-compatible mechanism) ϕ is a triple (M, G, θ) such that the message profile $\theta(v) = (\theta_i(v))_{i \in N} \in M$ is a Bayesian Nash equilibrium in (M, G) for each $v \in V$. We often write a profile $v \in V$ as (v_i, v_{-i}) to emphasize the role of buyer i, where $v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$. Similarly, $m = (m_i, m_{-i})$ and $\theta(v) = (\theta_i(v), \theta_{-i}(v))$, etc.

A mechanism is efficient if it always allocates the good to one of the bidders with the highest valuation; it is revenue-maximizing if it maximizes the revenue among possible all mechanisms:

Definition **1.** A mechanism (M, G, θ) is *efficient* if $\sum_{i \in N} x_i = 1$ and $x_i = 1 \implies v_i \geq v_j$ for all $i, j \in N$, for each $v \in V$ and each (x, p) in the support of $G(\theta(v))$.

Definition **2.** A mechanism (M, G, θ) is *revenue-maximizing* if its expected revenue given by $\mathbb{E}_{G\left((\theta_i(v_i))_{i \in N}\right)} \sum_{i \in N} p_i$ is weakly higher than the expected revenue in any other mechanism (M', G', θ').

[4]Our assumptions can be relaxed. First, the zero reservation price and buyers' valuations in $[0,1]$ assumptions are merely made for convenience; we can relax them while correspondingly changing the set of allowed bids in the definition of the first-price auction below. Second, we could allow an interdependent-value component in the utilities.

A mechanism is ex post individually rational if all buyers and the seller ex post benefit from participation:

Definition **3.** A mechanism (M, G, θ) is *ex post individually rational* if $u_i(x, p) \geq 0$ and $\sum_{j \in N} p_j \geq 0$ for each $i \in N$, each $v \in V$ and each (x, p) in the support of $G(\theta(v))$.

Notice that since at most one agent is assigned the good in any item allocation, the assumption of quasilinear utilities additionally implies that $x_i = 0 \implies p_i = 0$ for all $i \in N$ and all (x, p) in the support of $G(\theta(v))$; a losing buyer neither pays anything nor receives any positive transfer.

A mechanism is non-bossy if no buyer can change the outcome for any other buyer, by unilaterally changing her message, without changing her own outcome as well:

Definition **4.** A mechanism (M, G, θ) is *non-bossy* if, for all $v \in V$, all $i \in N$, and all $m_i \in M_i$, and any outcomes (x, p) and (x', p') in the support of $G(\theta(v))$ and $G(m_i, \theta_{-i}(v))$, respectively, we have that $(x_i, p_i) = (x'_i, p'_i) \implies (x, p) = (x', p')$.

Remark **1.** *While we focus on static mechanisms, all our results remain valid for dynamic mechanisms by an extension of the revelation principle. Indeed, suppose that M^D is an extensive-form game tree, G^D a distribution of outcome at terminal nodes, and θ^D a strategy profile. By the revelation principle, the allocation and payments implemented in a Bayesian-Nash equilibrium of the dynamic mechanism (M^D, G^D, θ^D) can be implemented in an equilibrium of a static direct mechanism. As all the above properties are defined in terms of outcomes, these properties remain well-defined for dynamic mechanisms. Furthermore, if the dynamic mechanism is efficient then so is the direct mechanism; if the dynamic mechanism is revenue-maximizing then so is the direct mechanism; if the dynamic mechanism is ex post individually rational then so is the direct mechanism; and if the dynamic mechanism is non-bossy then so is the direct mechanism.*

3 Main Results

We characterize the standard first-price auction. We first define this auction:

Definition **5.** Let $R \in [0, 1]$. A mechanism (M, G, θ) is a *first-price auction with reserve price R* if $M = [0, 1]$ and for any $v \in V$ with equilibrium message profile $\theta(v)$, and any (x, p) in the support of $G(\theta(v))$, we have: (i) $\sum\limits_{i \in N} x_i = 1$ if and only if $\max_{i \in N} \theta_i(v_i) \geq R$; (ii) $x_i = 1 \implies \theta_i(v_i) \geq \theta_j(v_j)$ for all $i, j \in N$; (iii) $x_i = 1 \implies p_i = \theta_i(v_i)$; and (iv) $x_j = 0 \implies p_j = 0$.

In a first-price auction, the set of messages for each buyer consists of real-valued bids, and the highest bidder wins the auction and pays her bid. In case of ties in the highest bids, winners are picked from this set with some probability. We denote the first-price auction mechanism by $\phi^{\text{FPA}} = (M^{\text{FPA}}, G^{\text{FPA}}, \theta^{\text{FPA}})$.

First-price auctions are ex-post individually rational and non-bossy; these two properties are straightforward. A first price auction is efficient iff $R = 0$; this observation follows from the standard analysis of the equilibrium of these auctions in symmetric private value settings, cf. e.g. Riley and Samuelson (1981).

We can preserve these properties of a first price auction even under certain non-essential modifications of its payment rule. First, we can have the winning bidder pay any injective function f of their bid provided the range of f is $[0, 1]$. Note the bijection between equilibria pre- and post- this transformation: if $\theta_i(v_i)$ are equilibrium bids before this transformation than $f^{-1}(\theta_i(v_i))$ are equilibrium bids after the transformation. In particular, the transformation has no impact on efficiency nor revenue of the seller.

Second, in case of a tie, we can have the winning bidder pay any amount $p_i \in [0, v_i x_i]$ as long as that amount does not depend on the bids of other bidders; the latter change of the payment rule do not affect non-bossiness and individual rationality, and, the conditioning event having zero probability, such change in the payment rule has no impact on the equilibrium bids and thus efficiency of the mechanism is also maintained. As ties have zero probability, the this transformation does not affect equilibrium bids and hence it does not affect efficiency nor expected revenue.

Our result shows that, other than the above two trivial degrees of freedom, the first price auctions are the sole mechanisms that are efficient, ex-post individually rational, and non-bossy. To capture the above degree of freedom we introduce a notion of equivalence of mechanisms; we could have alternatively embedded this payment rule freedom in the definition of the first price auction itself.

Definition **6.** Let $\phi = (M, G, \theta)$ and $\phi' = (M', G', \theta')$ be two mechanisms, and let $v \in V$ and $i \in N$ be arbitrary. We say ϕ is *equivalent* to ϕ' if, for any (x, p) in the support of $G(\theta(v))$ and any (x', p') in the support of $G'(\theta'(v))$: either (i) $x_i = x'_i$ and $p_i = p'_i$ for all bidders i; or (ii) there are two different bidders i and j such that $v_i = v_j \geq v_k$ for all $k \neq i, j$, $x_i = x'_j = 1$, $p_i, p'_j \in [0, v_i]$, the payment p_i does not depend on the bids of bidders different from i in ϕ, and the payment p'_j does not depend on the bids of bidders different from j in ϕ'. A mechanism satisfying certain properties is said to be *essentially unique* if all mechanisms satisfying these properties are equivalent to each other.

These definitions allow us to state our main result:

Theorem **1.** *A mechanism is equivalent to a first-price auction with zero reserve price if and only if it is non-bossy, efficient and ex-post individually rational.*

In other words, the first-price auction with zero reserve price is the essentially unique mechanism that is non-bossy, efficient and ex-post individually rational.

Proof: We showed above that all mechanisms equivalent to the first-price auctions are non-bossy, efficient, and ex-post individually rational. At the core of the result is the other direction of implication.

Consider any mechanism $\phi = (M, G, \theta)$ that is non-bossy, efficient and ex-post individually rational. Let $v \in V$ be a valuation profile. By Myerson (1981), efficiency and the assumption that $\theta(v)$ is a Bayes Nash equilibrium of (M, G) imply that buyers' interim utilities in ϕ are the same as in the first-price auction; interim utility is the expected utility of the buyer conditional on his or her type. We need to prove that the same obtains not only in expectation but also for each type profile of all buyers. Let (x, p) be an outcome in the support of $G(\theta(v))$. By efficiency, there is $i \in N$ with $v_i \geq v_j$ for all $j \neq i$ such that $x_i = 1$ and $x_j = 0$ for all $j \neq i$; that is i is the winning bidder. Ex post individual rationality and quasilinear utilities imply $p_j = 0$ for all $j \neq i$ and $p_i \in [0, v_i]$. Let J be the set of buyers such that $v_j \geq v_i$ for all $j \in J$ and all $i \in N$. These are the buyers with the highest valuations in v.

Case 1: $|J| = 1$. Then there is exactly one highest valuation in v; let us call the highest value buyer i. Furthermore, then the above described outcome is the unique outcome in the support of $G(\theta(v))$. Consider some $j \neq i$ and let (v'_j, v_{-j}) be such that $v'_j < v_i$. Then buyer i is still the unique highest value buyer and there is still a unique outcome (x', p') in the support of $G(\theta(v'_j, v_{-j}))$; in this unique outcome $x'_j = p'_j = 0$. Thus, non-bossiness implies that $(x, p) = (x', p')$. Moreover, for any $v'_j > v_i$ and (x'', p'') in the support of $G(\theta(v'_j, v_{-j})$, efficiency implies $x''_i = 0$, and ex post individual rationality implies $p''_i = 0$. Thus, in all valuation profiles with no ties at the highest valuation, the sole winner's (i's) payment depends only on whether she wins the good, and it is independent of other bidders' messages. The expected utility of bidder i is thus the probability of winning times the difference between v_i and p_i; as noticed above this expected utility is the same as in the first-price auction and by efficiency the probability of winning is also the same as in the first price auction. We can thus conclude that $p_i = \theta_i(v_i)$ and hence ϕ has the same outcome as the first price-auction.

Case 2: $|J| > 1$. Then the value vector v has at least two buyers with the highest valuation. Let (x, p) be an outcome in the support of $G(\theta(v))$ and let (x', p') be an outcome in the support of first price auction $G^{\text{FPA}}(\theta^{\text{FPA}}(v))$. The efficiency of ϕ implies that $x_j = 1$ for some $j \in J$ and the efficiency of the first price auction implies that $x'_{j'} = 1$ for some $j' \in J$. Ex post individual rationality implies that $p_i = 0$ for all $i \neq j$ and $p'_i = 0$ for all $i' \neq j'$, i.e., all losers' payments are zero, and moreover that the winners' payments lie in $[0, v_j]$ and $[0, v_{j'}]$, respectively. Moreover, by non-bossiness, payments for the winners do not depend on the bids of other buyers. This satisfies our equivalence condition for ϕ and ϕ^{FPA}, and so ϕ is equivalent to the standard FPA. ■

We may similarly characterize revenue-maximizing first-price auctions.

Theorem **2.** *A revenue-maximizing mechanism is equivalent to a first-price auction with a revenue-maximizing reserve price if and only if it is non-bossy and ex-post individually rational.*

In other words, the first-price auction with a revenue-maximizing reserve is the essentially unique revenue-maximizing mechanism that is non-bossy and ex-post individually rational.

A first-price auction with an optimal reserve price is revenue maximizing by Myerson (1981) and as discussed in our motivation of the equivalence definition, all equivalent mechanisms have the same revenue as the optimal first price auction. We also showed above that all mechanisms equivalent to the first-price auction are non-bossy and ex-post individually rational. The proof of the other direction of equivalence follows similar steps as the proof of our first theorem and is included in the appendix.

References

Adachi, T. and T. Kongo (2013): "First-price auctions on general preference domains: axiomatic characterizations," *Economic Theory Bulletin*, 1, 93–103.

Akbarpour, M. and S. Li (2020): "Credible Auctions: A Trilemma," *Econometrica*, 88, 595–618.

Hakimov, R. and M. Raghavan (2020): "Transparency in Centralised Allocation: Theory and Experiment," Working paper.

Myerson, R. B. (1981): "Optimal auction design," *Mathematics of Operations Research*, 6, 58–73.

Pápai, S. (2000): "Strategyproof Assignment by Hierarchical Exchange," *Econometrica*, 68, 1403–1433.

Pycia, M. and M. U. Ünver (2017): "Incentive-Compatible Allocation and Exchange of Discrete Resources," *Theoretical Economics*, 12, 287–329.

Raghavan, M. (2020): "Influence in Private-Goods Allocation," *Journal of Mathematical Economics*, 89, 14–28.

Riley, J. G. and W. F. Samuelson (1981): "Optimal Auctions," *American Economic Review*, 71, 381–392.

Root, J. and D. Ahn (2020): "Incentives and Efficiency in Constrained Allocation Mechanisms," Working paper.

Satterthwaite, M. and H. Sonnenschein (1981): "Strategy-proof allocation mechanisms at differentiable points," *Review of Economic Studies*, 48, 587–597.

Woodward, K. (2020): "Self-Auditable Auctions," Working paper.

A Proof of Theorem 2

Above we proved one implication of the theorem's equivalence. To prove the other implication, consider a mechanism $\phi = (M, G, \theta)$ that is revenue-maximizing, non-bossy, and ex-post individually rational. Let $v \in V$ be a valuation profile. By Myerson (1981), revenue-maximization and the assumption that $\theta(v)$ is a Bayes Nash equilibrium of (M, G) imply that buyers' interim utilities in ϕ are the same as in a first-price auction with reserve price $R \geq 0$; interim utility is the expected utility of the buyer conditional on his or her type. We need to prove that the same obtains not only in expectation but also for each type profile of all buyers.

Let (x, p) be an outcome in the support of $G(\theta(v))$. If $\max_{i \in N} v_i < R$ then Myerson (1981) implies that $x_i = 0$ for all $i \in N$ and hence ex post individual rationality implies that each bidder pays 0 and the claim of the theorem obtains. Consider the case $\max_{i \in N} v_i \geq R$. Then Myerson (1981) implies that the there is $i = \arg\max_{i \in N} v_i$ such that $x_i = 1$ and $x_j = 0$ for all $j \neq i$; that is i is the winning bidder. Ex post individual rationality and quasilinear utilities imply $p_j = 0$ for all $j \neq i$ and $p_i \in [0, v_i]$. Let J be the set of buyers such that $v_j \geq v_i$ for all $j \in J$ and all $i \in N$. These are the buyers with the highest valuations in v. As in Theorem 1, the reminder of the argument is split into two cases.

Case 1: $|J| = 1$. The argument in this case is analogous to the argument in the case $|J| = 1$ of Theorem 1, with revenue-maximization instead of efficiency implying that $x''_i = 0$ when (x'', p'') is the support of $G(\theta(v'_j, v_{-j})$ with $v'_j > v_i$.

Case 2: $|J| > 1$. The argument in this case is analogous to the argument in the case $|J| > 1$ of Theorem 1, with revenue-maximization and Myerson (1981), instead of efficiency, used to infer that $x_j = 1$ for some $j \in J$ and $x'_{j'} = 1$ for some $j' \in J$.