

The First-Price Principle of Maximizing Economic Objectives*

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Abstract

Allowing for reserve prices and any other restrictions on bid spaces, we show that the first-price auction format is sufficiently flexible to achieve any Bayesian-constrained standard objective, including maximizing or minimizing revenue, welfare, bidder surplus, and Gini mean difference as well as linear combinations of them. Furthermore, any Bayesian incentive compatible symmetric mechanism can be implemented by a convex combination of first-price auctions.

1 Introduction

First-price auctions have been used to sell objects throughout history. While it is known that the first-price auction format can be used to achieve objectives such as welfare maximization or revenue maximization, the auctioneer may be interested in other types of objectives. For instance, FTC uses consumer surplus as a standard, which translates to bidder surplus in the context of auctions. An auctioneer with distributional concerns may want to allocate the objects in a more equal manner by minimizing Gini mean difference. The government may wish to allocate objects to certain bidders with specific characteristics for political reasons, even if they are not willing to pay the most while wanting to increase the revenue at the

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same time. In this paper, we provide a class of objectives, which we call standard objectives, that includes the above examples and show the special role of first-price auctions in achieving standard objectives that include the examples above. A standard objective we study satisfies three conditions; anonymity, the convexity of the maximand set, and quasi-convexity.

Our main results show that any standard objective can be achieved by a first-price auction and that any symmetric mechanism is a lottery over first-price auctions. The second result can be restated as showing that the set of extreme points of symmetric mechanisms consist of first-price auctions. By showing that restricting attention to first-price auctions is sufficient for optimal design, we make the design problems tractable without the need to resolve to the revelation principle.

We prove our “first-price principle” in a general private values environment in which multiple buyers participate in a mechanism to win an object. Our environment incorporates as a special case the standard environment of Myerson (1981) and relaxes two of Myerson’s assumptions. We allow type distributions with arbitrary supports, finite or infinite, as opposed to requiring the support to be an interval; we also allow arbitrary type distributions and not only Lebesgue continuous ones.

Our focus on the first-price principle is at the source of our ability to relax these Myersonian assumptions. Unlike Myerson-inspired mechanism design analysis, we do not use incentive compatibility to pin the slope of participants’ mappings from types to payoffs; it is this slope determination that restricts Myersonian analysis to Lebesgue continuous distributions with interval supports. Instead, we treat the general type space as a limit case of discrete type spaces. We establish the first-price principle case for discrete type spaces and then show that it is preserved in any type space.¹ Our analysis of discrete type spaces blends in a new way analysis of interim allocations and Manelli and Vincent approach to Bayesian-Dominant-strategy allocation equivalence.

Our results contribute to the literature creating new tools for mechanism design. Most related is Kleiner, Moldovanu, and Strack (2021). Our and their papers were written independently and are complementary: they focus on characterizing optimal allocations while we characterize the space of optimal mechanisms. While they maintain the standard Myersonian continuity and support assumptions, our approach allows us to relax them.²

Earlier works on mechanism design with discrete type spaces focused on extending the slope determination behind revenue equivalence, and explored its limits, e.g. Lovejoy (2006).

¹In finite types space, we additionally show that lotteries over first-price auctions can achieve any anonymous objective whose maximand set is convex.

²Cf. An earlier work by Carbajal and Ely (2013) maintains the continuity assumptions but relaxes the indivisibility assumption. Nikzad (2022) maintains the continuity assumptions but considers the environment where the designer faces side constraints.

Beyond discrete space, the limits of the Myersonian slope approach was explored in Che and Gale (2006), who showed that first-price auctions yield higher expected revenue when risk-averse bidders are financially constrained, and in Chung and Olszewski (2007) and Heydenreich et al. (2009), who characterized the restrictions on the primitives that lead to the revenue equivalence result. Our work takes a different approach by allowing more restrictions on the bid space and shows that the first-price format dominates the second-price auction format in terms of the expected revenue.

By allowing a wide range of objective functions, our results are related to Condorelli (2013)—who shows that maximizing a linear combination of assignment values leads to assigning objects to agents with the highest willingness to pay—and Akbarpour, Dworzak, and Kominers (2020)—who shows that mechanisms maximizing a linear combination of revenue and welfare combine assortative (or market) allocation and uniform randomization. The objectives they study are standard in our sense and our Theorem 2 establishes analogous results for all standard objectives. Ours and their work rely on different methodologies: we establish our results for finite type spaces (in which revenue equivalence might fail) and extend them for more general type spaces, while the above three papers study continuum type spaces (in which revenue equivalence obtains).

2 Model

We study an environment with n bidders indexed by $i \in N = \{1, \dots, n\}$. Each bidder i has a utility type $\theta_i \in \Theta$, where Θ is a compact subset of \mathbb{R} . The discrete case $\Theta = \{\theta^1, \dots, \theta^T\}$ will play a special role in our proofs. We assume that each bidder's types are distributed according to distribution π and that the types are distributed i.i.d. across bidders. For brevity we also use π to refer to the joint distribution, and—in the discrete case—we write $\pi(\theta_1, \dots, \theta_n) = \pi(\theta_1) \dots \pi(\theta_n)$. In the discrete case, we index the types so that θ^t is strictly increasing in t .³

We study the allocation of a single indivisible object. A pure outcome is a vector $\mathbf{x} = (\mathbf{a}, \mathbf{p})$, where $\mathbf{a} = (a_1, \dots, a_n) \in \{0, 1\}^n$ such that $\sum_{i \in N} a_i = 1$ is the vector of allocations and $\mathbf{p} = (p_1, \dots, p_n) \in [\underline{p}, \infty)^n$ is the vector of payments for some $\underline{p} \leq 0$.⁴ When $a_i = 1$ and

³The strict monotonicity is without loss because we can represent the case when two types are equal, $\theta^t = \theta^{t+1}$, by using only one of these types and assigning it probability $\pi(\theta^t) + \pi(\theta^{t+1})$ for each profile of other bidders' types. In order to simplify exposition, when using superscripts (and subscripts) as above, we suppose that $\pi(\theta^{T+1})$ and similar undefined expressions take value 0.

⁴Bounding the possible payments from below means that we bound the potential subsidies the seller might offer to the bidders. We may interpret this bound as reflecting the seller participation or budget constraint. Imposing this is not needed when we study objectives such as revenue-maximization but it simplifies our analysis of more general objectives by ensuring the space of relevant mechanisms is compact.

$p_i = p$ we say that agent i receives the object and pays p . The space of pure outcomes is \mathbf{X} ; an individual outcome (a_i, p_i) of each bidder i belongs to $X = \{0, 1\} \times \mathbb{R}$. An outcome is a lottery over pure outcomes. Given a pure outcome $\mathbf{x} = (\mathbf{a}, \mathbf{p})$, the utility of each bidder i is given by $a_i \theta_i - p_i$; an agent utility from an outcome is the resulting expected utility. A social rule is a mapping from the space of type profiles Θ^N to the space of outcomes $\Delta(\mathbf{X})$. Given a social rule \mathbf{y} and a profile of types θ , we denote by $\mathbf{q}(\theta) = (q_i(\theta))_{i \in N}$ the vector of resulting expected allocations and by $\mathbf{t}(\theta) = (t_i(\theta))$ the vector of resulting expected payments, where each $q_i(\theta)$ is the expected value of a_i and $t_i(\theta)$ is the expected value of p_i calculated using the lottery $\mathbf{y}(\theta)$

Bidders participate in a mechanism that allocates a single indivisible object and collects payments from the bidders. A mechanism is given by the space of actions and a choice function. The space of actions is a set \mathcal{B} ; because auctions play a focal role in our analysis, we refer to actions $b \in \mathcal{B}$ as bids and to \mathcal{B} as the bid space. For the simplicity of exposition, we assume that \mathcal{B} is the same for all bidders; an assumption that is without loss of generality in the study of Bayesian Nash equilibria.⁵ A outcome function \mathbf{f} maps any profile of bids $\mathbf{b} = (b_i)_{i \in N}$ into a lottery $\mathbf{f}(\mathbf{b})$ over the space of pure outcomes \mathbf{X} ; using standard notation, we denote the space of the resulting lotteries by $\Delta(\mathbf{X})$. We say that a mechanism generates a social rule if there exists an equilibrium $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ such that $\mathbf{f}(\boldsymbol{\sigma}(\theta))$ is the same as the social rule applied to type profile θ for every profile in Θ^n . A social rule is implementable if there exists a mechanism that generates it and is individually rational if the expected utility of each bidder is weakly greater than 0 at any outcome in the range of the social rule.

A canonical example of a mechanism is a direct mechanism, in which $\mathcal{B} = \Theta$. We are also interested in auctions. An auction is a mechanism such that \mathcal{B} is a closed subset of $[p, \infty) \cup \{\emptyset\}$ and $\emptyset \in \mathcal{B}$, and the outcome function \mathbf{f} has the property that if at least one bidder submitted a bid in \mathbb{R} (a real bid) then for exactly one bidder i whose bid is weakly higher than all other submitted real bids $a_i = 1$, and for all other bidders $j \neq i$ the allocation $a_j = 0$; otherwise, for all bidders $a_i = 0$. A first-price auction is an auction such that $p_i = b_i$ if $a_i = 1$ and 0 otherwise, for any $\mathbf{x} \in \text{supp}(\mathbf{f}(\mathbf{b}))$ and $\mathbf{b} \in \mathcal{B}$. A second-price auction is an auction such that, for any $\mathbf{x} \in \text{supp}(\mathbf{f}(\mathbf{b}))$ and for any $\mathbf{b} \in \mathcal{B}^n$, if $a_i = 1$ then

$$p_i = \begin{cases} \min \mathcal{B} \cap \mathbb{R} & \text{if } i \text{ is the only bidder submitting a real bid} \\ \max\{b_j | j \in N - \{i\} \text{ and } b_j \neq \emptyset\} & \text{if some bidder } j \neq i \text{ submitted a real bid} \end{cases}$$

⁵Indeed, for any equilibrium in any mechanism in which each agent i has a space \mathcal{B}_i , we can find an outcome function and strategies in a mechanism in which each agent has the space $\mathcal{B} = \cup_{i \in N} \mathcal{B}_i$ such that the strategies are in equilibrium and the mechanism with action spaces all being \mathcal{B} implements the same mapping from types to outcomes as the original mechanism with action spaces \mathcal{B}_i .

and if $a_i = 0$ then $p_i = 0$. Note that $\min \mathcal{B} \cap \mathbb{R}$ is usually called the reserve price. Note also that in both first-price and second price auction, we allow the design of the bid space that goes beyond imposing a reserve price; in examples below we show that this additional degree of design freedom is necessary to achieve standard objectives.

Fix a social rule $\mathbf{y} = (y_1, \dots, y_n)$. Each bidder i with a type θ_i can compute the interim expected outcome (Q_i, T_i) by $\sum_{\theta_{-i} \in \Theta_{-i}} y_i(\theta_i, \theta_{-i}) \pi(\theta_i, \theta_{-i})$ and we say that Q_i is the interim expected allocation and T_i is the interim expected payment. We call $\mathbf{Q} = (Q_i)_{i \in N}$ the interim allocation rule and $\mathbf{T} = (T_i)_{i \in N}$ the interim payment rule. We call (\mathbf{Q}, \mathbf{T}) a reduced-form of \mathbf{y} and we also say that a mechanism $(\mathcal{B}, \mathbf{f})$ implements a reduced form mechanism (\mathbf{Q}, \mathbf{T}) if it generates a social rule \mathbf{y} such that (\mathbf{Q}, \mathbf{T}) is a reduced form of \mathbf{y} . Whenever we refer to a reduced-form mechanism, we always assume that there exists a mechanism $(\mathcal{B}, \mathbf{f})$ that implements it. We then say that an interim allocation rule \mathbf{Q} is feasible if there exists a reduced form mechanism (\mathbf{Q}, \mathbf{T}) and, a priori, we allow interim allocation rules that are not necessarily feasible. A reduced form mechanism (\mathbf{Q}, \mathbf{T}) is incentive compatible if $Q_i(\theta)\theta - T_i(\theta) \geq Q_i(\hat{\theta})\theta - T_i(\hat{\theta})$ for all $i \in N$ and $\theta, \hat{\theta} \in \Theta$; notice that the incentive compatibility requires that agents are willing to report their true types to the reduced form mechanism; in this sense the reduced form mechanism is a direct reduced form mechanism. A reduced form mechanism (\mathbf{Q}, \mathbf{T}) is individually rational if $Q_i(\theta)\theta - T_i(\theta) \geq 0$ for all $i \in N$ and $\theta, x \in \Theta$. An interim allocation rule \mathbf{Q} is symmetric if $Q_i(\theta) = Q_j(\theta)$ for all $i, j \in N$ and $\theta \in \Theta$. A reduced form mechanism (\mathbf{Q}, \mathbf{T}) is symmetric if \mathbf{Q} is symmetric and $T_i(\theta) = T_j(\theta)$ for all $i, j \in N$ and $\theta \in \Theta$. A social rule \mathbf{y} is called symmetric if $y_i(\boldsymbol{\theta}) = y_1(\tau^i(\boldsymbol{\theta}))$ for all i and $\boldsymbol{\theta}$ and transposition τ^i that interchanges the first and i -th coordinates of $\boldsymbol{\theta}$.

3 Objectives

The goal of the allocation is to maximize an objective such as welfare and revenue maximization or inequality minimization. Given π , an objective U_π is a mapping from the set of social rules to \mathbb{R} . We restrict attention to upper semicontinuous mappings so as to ensure that there is an implementable and individual rational social rule that maximizes the objective; note that the set of such rules is compact. We say that a social rule maximizes the objective if it is implementable, individually rational, and there is no other implementable and individually rational social rule that maps to a higher value. We say that a mechanism $(\mathcal{B}, \mathbf{f})$ maximizes the objective if it generates a social that maximizes the objective.

We say that an objective is standard if it satisfies the following four properties:⁶

(i) Anonymity: If \mathbf{y} maximizes the objective then so does $\tau^{-1} \circ \mathbf{y} \circ \tau$ for any bijection $\tau : N \rightarrow N$.

(ii) Convexity of the maximand set: If \mathbf{y}_1 and \mathbf{y}_2 maximize the objective then so does $\frac{1}{2}\mathbf{y}_1 + \frac{1}{2}\mathbf{y}_2$.

(iii) Quasi-convexity: For any symmetric \mathbf{y}_1 and \mathbf{y}_2 , if $\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2$ maximizes the objective for some $\lambda \in (0, 1)$, then either \mathbf{y}_1 or \mathbf{y}_2 (or both) maximize the objective.

(iv) Continuity: $U_\pi(\mathbf{y})$ is continuous in \mathbf{y} (in $L^1(\pi)$ metric on \mathbf{y}) and continuous in π (in Prokhorov metric on π).

(v) Revenue-monotonicity: $U_\pi(\mathbf{y})$ is increasing in $\sum_i t_i(\theta)$ for fixed $(q_i(\theta))_{i \in N}$.

Remark 1. The continuity and the revenue-monotonicity assumptions play a role in our analysis of general type spaces. In the special case of finite Θ , this assumption however can be completely relaxed. The revenue-monotonicity constraint can be relaxed if the distribution over types is atom-less. These additional restrictions beyond (i)-(iii) are not needed at any point of our analysis of the finite case.

In addition to the expected welfare and revenue maximization, the class of standard objectives includes seller's expected profit-maximization for any cost function of the seller, as well as linear combinations of expected welfare, revenue, and profits. Other examples of standard objectives include maximizing or minimizing the sum of payoffs of all bidders or of bidders of a specific type or the probability of that bidders with intermediate values win. Maximizing the weighted sum of payments, where weights depend on the bidders' types, is also a standard objective. In contrast, maximizing the variance of payments is an example of an objective that—in general—is not standard in the above sense because it violates our condition (ii).

Another interesting class of standard objectives are objectives minimizing inequality such as the Gini coefficient across bidders and their types

$$\text{Gini}_{BT} = \frac{\sum_{i,j} \sum_{\theta, \theta'} |Q_i(\theta) - Q_j(\theta')| \pi(\theta)\pi(\theta')}{\sum_{i,j} \sum_{\theta} Q_i(\theta)\pi(\theta)},$$

⁶See Bergson (1938) and Samuelson (1956) for the convexity assumptions; see May (1952), Sen (1970), and Pycia (2019) for anonymity. Because of upper semicontinuity, property (ii) implies that if \mathbf{y}_1 and \mathbf{y}_2 maximize the objective then so does $\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2$ for any $\lambda \in [0, 1]$. See Condorelli (2013) and Akbarpour, Dworzak, and Kominers (2020) for a case to go in our analysis beyond the revenue and standard welfare. The former paper study the maximization of a linear combination of type-dependent values the designer puts on assignments and the latter the maximization of type-dependent combination of revenue and welfare; both these objectives are standard in our sense.

as well as the ex ante Gini mean difference across bidders

$$\text{GMD}_B = \sum_{i,j} \sum_{\theta, \theta' \in \Theta} |Q_i(\theta) - Q_j(\theta')| \pi(\theta) \pi(\theta').$$

These objectives satisfy conditions (ii) and (iii) because they can be re-expressed as twice the sums over types $\theta > \theta'$; because the incentive compatibility implies the monotonicity of payoffs $U_i(\theta)$ in types θ , after such re-expression we can drop the absolute values in the formulas. Furthermore, maximizing any linear combinations (including with negative weights) of revenue, welfare, and the ex ante Gini mean difference is also standard.

4 The First-Price Principle

Our main result shows that first-price auctions are sufficient to implement all standard objectives. The key step of our analysis is showing that any symmetric reduced form mechanism that is incentive compatible and individually rational is equivalent to a lottery over first-price auctions and any such lottery is a symmetric mechanism (we exchangeably use the terms a lottery over mechanisms and a convex combination of the mechanisms). Establishing the decomposition insight for symmetric mechanism is sufficient to prove our opening claim because of the following:

Lemma 1. Any standard objective is maximized by a symmetric social rule.

Proof. Given a standard objective, suppose that a social rule \mathbf{y} maximizes the objective and let \mathcal{T} be the space of all permutations over the set of agents, N . Then the social rule $\frac{1}{|N|!} \sum_{\tau \in \mathcal{T}} \tau^{-1} \circ \mathbf{y} \circ \tau$ is symmetric, individually rational, and implementable. Furthermore, the properties (i) and (ii) of standard objectives ensure that this symmetric social rule maximizes the objective. \square

This lemma extends the straightforward observation that efficient allocations are symmetric and the classical insight that the allocations of revenue-maximizing mechanisms are symmetric as well. The latter insight goes back to Myerson (1981) who studied continuous type spaces; for discrete type spaces this insight was established by Lovejoy (2006) and Bergemann and Pesendorfer (2007).

The designer maximizing a standard objective can without loss restrict the bid space to bids chosen on equilibrium path. Furthermore, when a pure strategy equilibrium is played by bidders with finite type space, then a finite subset of bid space is used on equilibrium path. Hence, if we restrict attention to pure strategy equilibria then considering finite bid

spaces is sufficient. A similar claim for mixed strategies relies on a more subtle argument and we omit it because we are not relying on it.

Our main results are the following

Theorem 1. For finite type spaces and any anonymous objective with convex maximand set, there is a lottery over first-price auctions that maximizes this objective.

By additionally requiring that the objective is quasi-convex on the set of symmetric social rules, we obtain

Theorem 2. For any standard objective, there exists a bids space \mathcal{B} such that a first-price auction with \mathcal{B} maximizes the objective.

This result implies that, for standard objectives, the allocation is piecewise assortative, and it is uniformly random on bunched intervals, that is monotonic partitions.⁷

Corollary 1. For all standard objectives, there is a monotone partition of the type space such that the object is assigned to an agent in the highest element of the partition. Furthermore, the elements of the partition are of two types: for some elements the object is assigned uniformly at random to all bidders with types in the element; for other elements the object is assigned uniformly at random to bidders with the highest type.

In particular, this corollary contains the main insight of Akbarpour, Dworzak, and Kominers (2020). In addition, it implies similar insights for other standard objectives, e.g. for the maximization of any linear combination of revenue, welfare, and the ex ante Gini mean difference.

In the appendix, we initially prove Theorems 1 and 2 for finite type spaces, building on the following Theorem 3. We then extend Theorem 2 to general type spaces by approximating them with finite type spaces. We use Helly’s selection theorem to select a convergent sequence of first-price auctions that are optimal a sequence of finite type spaces and we show that the limit of this sequence is a first-price auction that maximizes the objective.

Theorem 3. For finite type spaces, every symmetric, incentive compatible, and individually rational reduced-form mechanism (\mathbf{Q}, \mathbf{T}) is a reduced-form of a lottery over first-price auctions.

We devote the next section to establishing Theorem 3 as part of our general analysis of extreme points in the space of all mechanisms (see Theorem 6); in particular we show there that first-price auctions are such extreme points (Theorem 5).

⁷Monotonic partitions are defined in Section 6.

Proof of Theorems 1 and 2. By Lemma 1, we can restrict our attention to symmetric mechanisms. Note that the space of implementable and individually rational social rules is convex and compact. Theorem 1 then follows directly from Theorem 3. To establish Theorem 2 notice further that, by the quasi-convexity of the standard objective, the standard objective is maximized by a mechanism $(\mathcal{B}^*, \mathbf{f}^*)$ that generates \mathbf{y}^* which is an extreme point of the space. Note that the set \mathbf{S} of symmetric, incentive compatible, and individually rational reduced form mechanisms is convex and compact. Denote the reduced form mechanism of \mathbf{y}^* by $(\mathbf{Q}^*, \mathbf{T}^*)$. Then, $(\mathbf{Q}^*, \mathbf{T}^*)$ is also at an extreme point of the set \mathbf{S} . By Theorem 3, any extreme points of \mathbf{S} is a reduced form of a first-price auction, so is $(\mathbf{Q}^*, \mathbf{T}^*)$. \square

An analogue of Theorem 2 holds true for second-price auctions even those that do not maximize any standard objective.

Theorem 4. Any expected allocation and payment in a symmetric equilibrium in a second-price auction can be implemented by a first-price auction.

Proof. Given a symmetric equilibrium strategy σ_{SPA} in the second-price auction with the bid space \mathcal{B}_{SPA} , we use $Q^{SPA}(b)$ and $T^{SPA}(b)$ to denote the expected winning probability and the expected payment of a bidder when he plays a bid $b \in \mathcal{B}_{SPA}$ assuming that all the other bidders follow the strategy σ_{SPA} . Without loss of generality, let us assume that for any $b \in \mathcal{B}_{SPA}$, there exists $\theta \in \Theta$ such that $b \in \sigma_{SPA}(\theta)$.⁸ Define a mapping $X : \mathcal{B}_{SPA} \rightarrow \mathbb{R} \cup \{\emptyset\}$ by

$$X(b) = \begin{cases} \frac{T^{SPA}(b)}{Q^{SPA}(b)} & \text{if } b \neq \emptyset \\ \emptyset & \text{if } b = \emptyset \end{cases}.$$

We construct the bid space for a first-price auction and a strategy; $\mathcal{B}_{FPA} = \{X(b) | b \in \mathcal{B}_{SPA}\} \cup \{\emptyset\}$ and $\sigma_{FPA}(\theta) = X(\sigma_{SPA}(\theta))$.⁹ Denote the expected winning probability and the expected payment of a bidder playing $b \in \mathcal{B}_{FPA}$ by $Q^{FPA}(b)$ and $T^{FPA}(b)$ assuming that the other bidders follow the strategy σ_{FPA} . To show that σ_{FPA} is a symmetric equilibrium strategy in the first-price auction, we only need to check that the mapping $X(b)$ is strictly increasing in b for all $b \in \mathcal{B}_{SPA} - \{\emptyset\}$ as then $Q^{FPA}(X(b)) = Q^{SPA}(b)$ and $T^{FPA}(X(b)) = T^{SPA}(b)$ for all $b \in \mathcal{B}_{SPA}$, thus the incentive properties of the second price auction are inherited by the first price auction we construct.¹⁰

For the monotonicity verification, take any two arbitrary b and b' in $\mathcal{B}_{SPA} - \{\emptyset\}$ such that $b' > b$. Then

⁸Otherwise, we can use a subset of \mathcal{B}_{SPA} to construct a bid space and an symmetric equilibrium in the first-price auction.

⁹With a slight abuse of notation, $X(\sigma_{SPA}(\theta))$ denotes the mixed strategy, where the probability of playing a bid $b \in \sigma_{SPA}(\theta)$ is equal to the the probability of playing $X(b)$ in the mixed strategy σ_{FPA} .

¹⁰The weak monotonicity would not be sufficient for this claim because of the possible tie-breaking.

$$Q^{SPA}(b)\sigma_{SPA}^{-1}(b) - T^{SPA}(b) \geq Q^{SPA}(b')\sigma_{SPA}^{-1}(b) - T^{SPA}(b'),$$

which is equivalent to

$$\frac{Q^{SPA}(b)}{Q^{SPA}(b')} [\sigma_{SPA}^{-1}(b) - \frac{T^{SPA}(b)}{Q^{SPA}(b)}] \geq \sigma_{SPA}^{-1}(b) - \frac{T^{SPA}(b')}{Q^{SPA}(b')}.$$

Because $0 < Q^{SPA}(b) < Q^{SPA}(b')$, we can conclude that

$$\frac{T^{SPA}(b')}{Q^{SPA}(b')} > \frac{T^{SPA}(b)}{Q^{SPA}(b)},$$

thus $X(\cdot)$ is strictly increasing as required. \square

Is the converse true as well? Can we implement the outcomes of first-price auctions in second-price auctions? In Proposition 1, we show that this is so when there are at most two types, but, in general, the answer is negative as Example 1 illustrates.

Proposition 1. Any expected allocation and payment in a pure symmetric equilibrium in a first-price auction can be implemented by a second-price auctions if $|\Theta| \leq 2$.

Proof. Suppose that $\Theta = \{\theta_L, \theta_H\}$ and $\theta_H > \theta_L$. Denote the probability that a bidder has a type θ_H by π . Given a symmetric equilibrium in the first-price auction, denote the expected allocation and the expected payment for a bidder with type θ by $Q(\theta)$ and $T(\theta)$. Note that $Q(\theta_H) \geq Q(\theta_L)$. If $Q(\theta_L) = Q(\theta_H) = 0$, define $\mathcal{B} = \{b, \emptyset\}$, where b is any number strictly greater than θ_H . Then, the only equilibrium strategy in the second-price auction with \mathcal{B} is to choose \emptyset . If $Q(\theta_L) = 0$ and $Q(\theta_H) > 0$, define $\mathcal{B} = \{b, \emptyset\}$, where $b = \frac{T(\theta_H)}{Q(\theta_H)}$. Then, it is a symmetric equilibrium that the bidders with the higher type choose b and the bidders with the lower type choose \emptyset . If $Q(\theta_H) = Q(\theta_L) > 0$, define $\mathcal{B} = \{b, \emptyset\}$, where $b = \frac{T(\theta_H)}{Q(\theta_H)}$. Then, it is a symmetric equilibrium that all the bidders choose b . Suppose that $Q(\theta_H) > Q(\theta_L) > 0$. Define $\mathcal{B} = \{b_L, b_H, \emptyset\}$, where $b_L = \frac{T(\theta_L)}{Q(\theta_L)}$ and $b_H = \frac{T(\theta_H) - (n-1)T(\theta_L)}{Q(\theta_H) - (n-1)Q(\theta_L)}$ and consider a strategy where the bidders with the lower type chooses b_L and the bidders with the higher type choose b_H . We show that (i) b_H is well-defined, (ii) $b_H > b_L$, and (iii) it is an equilibrium that the bidders with the lower type choose b_L and the bidders with the higher type choose b_H .

(i) Note that $Q(\theta_H) = \sum_{m=0}^{n-1} \frac{1}{m+1} \binom{n-1}{m} \pi^m (1-\pi)^{n-m-1} = \sum_{m=1}^{n-1} \frac{1}{m+1} \binom{n-1}{m} \pi^m (1-\pi)^{n-m-1} + (1-\pi)^{n-1}$ and $Q(\theta_L) = \frac{1}{n-1} (1-\pi)^{n-1}$. Therefore, $Q(\theta_H) > (n-1)Q(\theta_L)$.

(ii) Note that $[Q(\theta_H) - (n-1)Q(\theta_L)] * b_L + (n-1)T(\theta_L) = Q(\theta_H)T(\theta_L)/Q(\theta_L) < T(\theta_H)$, which implies $b_L < b_H$.

(iii) It is trivial that the expected payment of a low-type bidder in the given strategy in the second-price auction is $T(\theta_L)$. For a bidder with the type θ_H , he pays b_L only when all the other bidders have the lower type. Therefore, the expected payment of a bidder with the higher type is $(1 - \pi)^{n-1}b_L + (Q(\theta_H) - (1 - \pi)^{n-1})b_H = T(\theta_H)$. Because the expected winning probability and the expected payment for each type in the suggested strategy in the second-price auction coincide with the ones in the first-price auction, the suggested strategy is indeed an equilibrium strategy. \square

In particular, the above proposition shows that with two types, an analogue of revenue equivalence obtains: same objectives can be achieved by first-price and second-price auctions. With three or more types, the revenue equivalence fails and the following example shows that revenue in a first-price auction might be strictly higher than the maximum revenue achievable in second-price auctions.

Example 1. There are 2 bidders, and each bidder i 's value θ_i is drawn independently following

$$\theta_i = \begin{cases} 1 & \text{with probability 0.6,} \\ 11/4 & \text{with probability 0.2,} \\ 13/2 & \text{with probability 0.2.} \end{cases}$$

Suppose $\mathcal{B} = \{1, 2, 3\}$. Then,

$$\sigma(\theta_i) = \begin{cases} 1 & \text{if } \theta_i = 1, \\ 2 & \text{if } \theta_i = 11/4, \\ 3 & \text{if } \theta_i = 13/2. \end{cases}$$

is a symmetric equilibrium of the first-price auction with designed bids \mathcal{B} and

$$Q(\theta) = \begin{cases} 0.6 & \text{if } \theta = 1, \\ 0.8 & \text{if } \theta = 11/4, \\ 1 & \text{if } \theta = 13/2. \end{cases}$$

The expected payment for each type is

$$T(1) = 0.3, \quad T(11/4) = 1.4, \quad T(13/2) = 2.7.$$

Now, consider a second-price auction to check if there exists a bids space such that the second-price auction achieves the same outcome with. Since the lowest type's utility in the

equilibrium is 0, the equilibrium bid level for the lowest type is 1. The expected payment for the middle type when he places a bid level b_M is

$$b_L \times 0.6 + b_M \times \frac{1}{2} \times 0.2 = 0.6 + b_M \times \frac{1}{2} \times 0.2.$$

In order to match the expected payment under the second-price auction to the one in the first-price auction, b_M must solve the equation

$$0.6 + b_M \times \frac{1}{2} \times 0.2 = 1.4$$

and $b_M = 8$. We cannot separate the highest type from the middle type because a bidder with the value $13/2$ will never place a bid above 8 in a symmetric equilibrium.

The idea behind of the example is as follows. In a second price auction with designed bids, the designer can make a bidder to place a bid above their own value. If the bidder pays the second highest bid with very high probability, the level of bid the bidder is willing to place can be arbitrarily large. In the example, the bid level that a middle type is willing to place is even above the value of the high type and it makes the high type deviates from the bid he was suppose to submit.

5 Application: Credible Implementation

We now study that how the auctioneer can use a first-price auction to achieve his goal credibly in the sense of Akbarpour and Li (2020). Mapping their terminology to our study of auctions, we say that an observation of bidder i is (a_i, p_i) , where, as above, p_i is the payment of bidder i and $a_i = 1$ iff bidder i wins, and it equals 0 otherwise. Given an allocation rule (\mathbf{a}, \mathbf{p}) and equilibrium σ , another allocation rule $(\hat{\mathbf{a}}, \hat{\mathbf{p}})$ is safe if for every bidder i and every profile of types $(\theta_1, \dots, \theta_n)$, the observation of i given the allocation $(\hat{\mathbf{a}}, \hat{\mathbf{p}}) \sigma(\theta_1, \dots, \theta_n)$ is the same as the observation of this bidder given allocation $(\mathbf{a}, \mathbf{p}) \sigma(\theta_i, \theta'_{-i})$ for some profile of types θ'_{-i} of bidders other than i . A first-price auction is credible if no safe allocation rule may strictly improve the objective at some profile of types. Akbarpour and Li (2020) showed that first-price auctions are credible when the objective is a weighted average of revenue and welfare; using their approach, one can easily show a slightly more general result.

Lemma 2. Suppose that $\Theta \subset \mathbb{R}_+$ and the bidders play a symmetric equilibrium. If a standard objective is a function of revenue and social welfare, and it is weakly increasing in each argument, then any first-price auctions with \mathcal{B} such that $\mathcal{B} \cap \mathbb{R} \subseteq \mathbb{R}_+$ is credible.

The restriction on the bid space $\mathcal{B} \cap \mathbb{R} \subseteq \mathbb{R}_+$ matters as in its absence there might be a safe deviation in which the item is not allocated when all bids are negative.

Proof. Denote the winner by i in a given first-price auction by following the rule. Suppose that the auctioneer safely deviate by picking a winner j , which is not i . If $b_j = b_i$, then it is easy to see that the revenue does not change. Moreover, the social welfare does not change either since the auctioneer cannot distinguish the types among the bidders who bid the same. If $b_j < b_i$, then the revenue decreases and the social welfare does not increase since any symmetric equilibrium is weakly monotonic. The safe deviation that does not sell the item does not increase the objective because we assume that the types and real bids are positive. \square

This lemma and our 2 imply the following.

Corollary 2. If a standard objective is a function of revenue and social welfare, and it is increasing in each argument, then this objective can be credibly achieved by a first-price auction.

Proof. By 2, a first-price auction maximizes the objective. We can restrict our attention to the environment where $\Theta \in \mathbb{R}_+$ and only positive bids are allowed without loss of generality, because if a bidder with a negative type wins the item and pays a negative bid, the auctioneer can increase the objective by setting $\min \mathcal{B} \cap \mathbb{R}$ to be non-negative. Then, the corollary is a direct implication of 2. \square

While one may conjecture that the above results can extend to any standard objectives, the following example shows that some standard objectives cannot be credibly achieved by a first-price auction.

Example 2. There are two bidders and each bidder's type is drawn from $\Theta = \{0, 1\}$ uniformly at random. While the auctioneer prefers more revenue, he also has a distributional concern. Specifically, the objective of the auctioneer is $\mathbb{E}_\theta[t_1(\theta) + t_2(\theta)]$ if a bidder whose type is 1 wins the item, and $\mathbb{E}_\theta[t_1(\theta) + t_2(\theta)] + A$ if a bidder whose type is 0 wins the item. One can verify that, with $A \in [1, 2]$, the first-price auction with $\mathcal{B} = \{0, \frac{2}{3}\}$ maximizes the objective with the symmetric equilibrium $\sigma(0) = 0$ and $\sigma(1) = \frac{2}{3}$ and the resulting utility of the auctioneer is $\frac{1}{2} + \frac{1}{4}A$. Suppose that auctioneer deviates by allocating the item to a bidder that bid 0 when the other bidder bid 1. Note that this deviation is safe and the resulting utility is $\frac{1}{6} + \frac{3}{4}A$, which is strictly greater than $\frac{1}{2} + \frac{1}{4}A$ for $A > \frac{2}{3}$.

6 Extreme Points of Reduced Form Direct Mechanisms with Finite Type Spaces

In this auxiliary section, we restrict attention to finite type spaces and study the set of incentive compatible symmetric social rules and show that each extreme point of the set takes a special form that can be generated by a first-price auction. A natural class of symmetric social rule allocates the object with equal probabilities to the bidders with the highest type. We generalize this class by allowing equal-probability allocation to bidders in a highest interval of types (which call the highest tier) as follows. We say that a partition $\Theta_1, \Theta_2, \dots, \Theta_J$ of Θ is monotonic partition if $\max \Theta_j < \min \Theta_{j+1}$ for all $j = 1, \dots, J-1$ and we say that a social rule \mathbf{y} is monotonic hierarchical if there exists a monotonic partition $\Theta_1, \Theta_2, \dots, \Theta_J$ of Θ such that for all i , one of the following two possibilities obtains:

$$q_i(\theta_1, \dots, \theta_n) = \begin{cases} \frac{1}{|\{l: \theta_l \in \Theta_j\}|}, & \text{if } \theta_i \in \Theta_j \text{ and } \theta_{i'} \notin \cup_{l=j+1}^J \Theta_l \text{ for all } i' \in N, \\ 0, & \text{otherwise,} \end{cases}$$

or

$$q_i(\theta_1, \dots, \theta_n) = \begin{cases} \frac{1}{|\{n: \theta_n \in \Theta_j\}|}, & \text{if } j \geq 2, \theta_i \in \Theta_j \text{ and } \theta_{i'} \notin \cup_{l=j+1}^J \Theta_l \text{ for all } i' \in N, \\ 0, & \text{otherwise.} \end{cases}$$

The former possibility encompasses social rules that always allocate the object, while the latter possibility encompasses social rules in which the object is unallocated if no bidder's type is above a certain threshold. The popular implementation of the latter of social rules is via a reserve price. More generally, in the appendix, we prove the following:

Theorem 5. Any implementable, individually rational, and monotonic hierarchical social rule is generated by a first-price auction.

This theorem, together with the decomposition result we state next, allows us to prove the key results of Section 3.

Theorem 6. Any implementable and individually rational social rule is a convex combination of implementable, individually rational, and monotonic hierarchical social rules.

In particular, this theorem implies that if a social rule \mathbf{y} is an extreme point of the space of symmetric, implementable, and individually rational social rules then \mathbf{y} is monotonic hierarchical. Theorem 6 extends an insight of Manelli and Vincent (2010), who studied continuous type spaces. We establish this theorem as immediate consequence of the following

two lemmas, proven in the appendix. In proving the first lemma—but not the second—we are able to leverage the approach that Manelli and Vincent (2010) employed in their analysis.¹¹

In order to prove the theorem 6, we study the space, denoted by \mathcal{Q} of non-decreasing mappings from Θ to $[0, 1]$, where each mapping also satisfies the condition in the lemma 8. The first lemma shows that any extreme point of the space is an interim expected allocation of a monotonic hierarchical social rule.

Lemma 3. The following three conditions are equivalent:

- (i) Q is an extreme point of \mathcal{Q} .
- (ii) There is a monotonic hierarchical rule \mathbf{y} such that $(Q, \dots, Q) \in \mathcal{Q}^n$ is an interim allocation rule of \mathbf{y} .
- (iii) The coarsest monotonic partition $\{\Theta_1, \dots, \Theta_J\}$ of Θ that is adapted to Q satisfies

$$Q(\Theta_j) = \frac{[\sum_{l=1}^j \pi(\Theta_l)]^n - [\sum_{l=1}^{j-1} \pi(\Theta_l)]^n}{n\pi(\Theta_j)}, \text{ for } j = 2, \dots, J, \quad (1)$$

$$Q(\Theta_1) = 0 \text{ or } \frac{\pi(\Theta_1)^{n-1}}{n}. \quad (2)$$

This characterization of extreme points implies that, in our environment, any symmetric interim allocation rule is a convex combination of interim allocation rules of hierarchical social rules. To extend this insight to mechanisms, that is allocation rules and payment rules, we use Farkas Alternative to prove the following decomposition lemma. This lemma has no direct counterpart in Manelli and Vincent (2010), who focus on allocations.¹²

Lemma 4. (Decomposition Lemma) Consider a reduced-form $(\mathbf{Q}, \mathbf{T}) = ((Q_1, T_1), \dots, (Q_n, T_n))$ of a social rule that is incentive compatible, individually rational. Suppose that $Q_i = \sum_{k=1}^K \alpha_k Q_i^k$ for some Q_i^k 's and weights $\alpha_k > 0$ such that $\sum_{k=1}^K \alpha_k = 1$, for all $i \in N$. Then, $Q_i^1(\theta), \dots, Q_i^K(\theta)$ are non-decreasing in θ if and only if there exist T_i^1, \dots, T_i^K such that (Q_i^k, T_i^k) is incentive compatible, individually rational, and $T_i = \sum_{k=1}^K \alpha_k T_i^k$.

¹¹Earlier developments of Manelli and Vincent's structure theorem, most notably Gershkov et al. (2013), analyzed discrete type spaces, but their approach was different from ours; in particular, they did not characterize incentive-compatible mechanisms in terms of convex combinations of simpler classes of rules, and neither Lemma 3 nor Lemma 4 has a counterpart in their analysis.

¹²As they observe, in the continuous environment they study, the payment functions can be reconstructed, up to a constant, from incentive compatibility conditions. In our discrete environment such a recovery is not possible.

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A Proofs

We rely on two well-known results from linear programming.

Lemma 5. (Farkas’ Alternative)¹³ Suppose that $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{l \times n}$, and $d \in \mathbb{R}^l$. Then exactly one of the following two statements is true:

¹³See Border (2015) for the this version of Farkas’ Alternative.

- There exists $x \in \mathbb{R}^n$ such that $Cx = d$ and $Ax \leq b$.
- There exist $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^m$ such that $zA + yC = 0$, $z \geq 0$, and $z \cdot b + y \cdot d < 0$.

Lemma 6. (Fundamental Theorem of Linear Programming)¹⁴ Suppose that $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Consider a set $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. Then, $x^* \in P$ is an extreme point of P if and only if $Ax^* \leq b$ has n binding constraints that are linearly independent.

We also rely on the following well-known results in mechanism design.

Lemma 7. (Characterizations of Incentive Compatibility)

- (Spence-Mirrlees Condition)¹⁵ For every interim probability rule \mathbf{Q} , there exists an interim payment rule \mathbf{T} such that the reduced-form mechanism (\mathbf{Q}, \mathbf{T}) is incentive compatible if and only if \mathbf{Q} is non-decreasing.
- (Local IC guarantees Global IC)¹⁶ The reduced-form mechanism (\mathbf{Q}, \mathbf{T}) is incentive compatible and individually rational if and only if for all $i \in N$,

$$Q_i(\theta^t)\theta^t - T_i(\theta^t) \geq Q_i(\theta^{t-1})\theta^t - T_i(\theta^{t-1}), \text{ for all } t = 1, \dots, T, \quad (3)$$

$$Q_i(\theta^{t-1})\theta^{t-1} - T_i(\theta^{t-1}) \geq Q_i(\theta^t)\theta^{t-1} - T_i(\theta^t), \text{ for all } t = 2, \dots, T. \quad (4)$$

Because we assumed that $Q_i(\theta^0) = 0$ and $T_i(\theta^0) = 0$ (cf. footnote 3), the condition (3) encompasses individual rationality for type θ^1 .

Lemma 8. (Maskin-Riley-Matthews-Border Condition)¹⁷ A symmetric interim allocation rule $\mathbf{Q} = (Q, \dots, Q)$ is feasible if and only if

$$n \sum_{\{\theta \in \Theta: Q(\theta) \geq \alpha\}} Q(\theta)\pi(\theta) \leq 1 - \left(\sum_{\{\theta \in \Theta: Q(\theta) < \alpha\}} \pi(\theta) \right)^n, \text{ for all } \alpha \in [0, 1]. \quad (5)$$

Then, the above lemmas tell us that

$$\mathcal{Q}^S = \{(Q, \dots, Q) : \Theta^n \rightarrow [0, 1]^n | Q \text{ satisfies (5) and non-decreasing}\}. \quad (6)$$

¹⁴See Vohra (2005).

¹⁵For more detailed discussions, see Spence (1974), Mirrlees (1976), and Rochet (1987).

¹⁶For more detailed discussions, see McAfee and McMillan (1988), Carroll (2012), and Mishra, Pramanik, and Roy (2016). The equivalence of local and global incentive compatibility conditions relies on the single crossing property, which is satisfied in our setting.

¹⁷Maskin and Riley (1984) studied the condition when the type space is continuous and the allocation is a step function in their analysis of optimal auctions with risk-averse bidders. Matthews (1984) showed that the condition holds for any increasing allocation functions. Border (1991) proved the condition for general type spaces and allocation functions.

Because the type space is finite, for any $(Q, \dots, Q) \in \mathcal{Q}^S$ there is a partition $\{\Theta_1, \dots, \Theta_J\}$ of Θ such that $Q(\theta) = Q(\theta')$ for all $\theta, \theta' \in \Theta_j$. We then say that the partition is adapted to Q and write $Q(\Theta_j)$ to denote the common value $Q(\theta)$ of all $\theta \in \Theta_j$.

Proof of Theorem 2

For finite type spaces, it is a direct implication of Theorem 3. Let us consider the general case of Θ being a compact subset of \mathbb{R} . The linear combinations of Dirac measures are dense in the measure space on Θ equipped with Weak-*Topology.¹⁸ Thus, there exists a sequence $(\mu_k)_{k \in \mathbb{N}}$ of probability measures with finite supports that converges to μ ; we denote by Θ_k the support of μ_k . Because the set of atoms of μ is at most countable, we can ensure that for any atom θ in μ there is $k(\theta)$ such that for all $k \geq k(\theta)$ we have $\mu_k(\theta) = \mu(\theta)$; in particular, μ_k then also has an atom at θ . By the discrete version of Theorem 2, for each μ_k , there is FPA with bid space B_k and symmetric equilibrium bid function $\tilde{b}_k : \Theta_k \rightarrow B_k$ that implements a social rule which maximizes the objective under μ_k . By incentive compatibility of the equilibrium bid, \tilde{b}_k is weakly increasing. We extend the domain of \tilde{b}_k to Θ by defining

$$b_k(\theta) \in \arg \max_{b \in B_k} P_k(b)(\theta - b),$$

where $P_k(b) = \sum_{n=1}^N \frac{1}{n} \binom{N-1}{n-1} \left(\mu_k(\{\theta | b > \tilde{b}_k(\theta)\}) \right)^{N-n} \left(\mu_k(\{\theta | b = \tilde{b}_k(\theta)\}) \right)^{n-1}$ is the probability of winning against Θ_k bidders bidding \tilde{b}_k . By incentive compatibility of \tilde{b}_k , for all $\theta \in \Theta_k$ we can select $b_k(\theta)$ so that $b_k(\theta) = \tilde{b}_k(\theta)$. We can further select b_k that is weakly increasing because if a type θ bidder weakly prefers b to b' , where $b > b'$, then any type above θ weakly prefers b to b' .

By construction $b_k(\cdot)$ is weakly increasing and uniformly bounded, thus, we can apply Helly's selection theorem to conclude that the sequence of functions $(b_k)_{k \in \mathbb{N}}$ admits a pointwise convergent subsequence. Let b be the limit of this subsequence; note that b is also weakly increasing. Without of generality, in the sequel we assume that $(b_k)_{k \in \mathbb{N}}$ itself is this convergent subsequence.

In order to show that $b(\cdot)$ is a bidding strategy in a symmetric equilibrium in the limit, notice that it is enough to show that $P_k(b_k(\theta)) \rightarrow P(b(\theta))$ for all θ , where $P(b(\theta))$ denotes the winning probability by bidding $b(\theta)$ when all the other bidders follow the same bidding strategy $b(\cdot)$ under the probability measure μ . We show that $\mu_k(\{x | b_k(x) < b_k(\theta)\})$ and $\mu_k(\{x | b_k(x) = b_k(\theta)\})$ converge to $\mu(\{x | b(x) < b(\theta)\})$ and $\mu(\{x | b(x) = b(\theta)\})$, which is followed by the convergence of $P_k(b_k(\theta))$ to $P(b(\theta))$.

¹⁸See Example 8.1.6 (i) in Bogachev (2006) for the proof.

Claim 1. We have $\mu_k(I) \rightarrow \mu(I)$ for subset $I \subseteq \Theta$ if the boundary ∂I is countable.

If I itself is countable, then the convergence is directly implied by the construction of μ_k . Suppose that I is uncountable and ∂I is countable. Notice that $\liminf \mu_k(I) \geq \liminf \mu_k(I - (I \cap \partial I)) + \liminf \mu_k(I \cap \partial I)$. By the Portmanteau theorem, $\liminf \mu_k(I - (I \cap \partial I)) \geq \mu(I - (I \cap \partial I))$. Because $I \cap \partial I$ is countable, $\liminf \mu_k(I \cap \partial I) = \mu(I \cap \partial I)$. Combining these establishes $\liminf \mu_k(I) \geq \mu(I)$.

Similarly, $\limsup \mu_k(I) \leq \limsup \mu_k(I \cup (\partial I - I)) + \limsup [-\mu_k((\partial I - I))]$. By applying the Portmanteau theorem and using the convergence of $\mu_k((\partial I - I))$, it follows that $\limsup \mu_k(I) \leq \mu(I)$. Therefore, $\lim_k \mu_k(I) = \mu(I)$ if ∂I is countable.

Claim 2. For any $\theta \in \Theta$, $\limsup_k I_k(\theta) \subseteq I(\theta)$.

Proof of Claim 2: Suppose that $x \in \limsup_k I_k(\theta)$. Because $b_k(x) = b_k(\theta)$ infinitely often and $b_k(\cdot)$ converges to $b(\cdot)$, $b(x) = b(\theta)$, thus, $x \in I(\theta)$.

Claim 3. Suppose that $\mu(I(\theta)) > 0$ and for all $x \in I(\theta)$, $b(x) < x$. Then, $I(\theta) \subseteq \liminf_k I_k(\theta)$.

Proof of Claim 3: Suppose that there exists $\theta' \in I(\theta) - \liminf_k I_k(\theta)$. Notice that $\theta' \neq \theta$ because $\liminf_k I_k(\theta)$ includes θ . For any K , there exists $k > K$ such that $b_k(\theta') \neq b_k(\theta)$. The monotonicity of b_k allows us to pick $x, y \in I(\theta)$ such that for any K , there exists $k > K$ such that $b_k(x) < b_k(y)$ and $\mu([x, y]) > 0$.

From the incentive compatibility for x , the following holds;

$$[P_k(b_k(y)) - P_k(b_k(x))] (x - b_k(x)) \leq P_k(b_k(y)) (b_k(y) - b_k(x)).$$

By the monotonicity of $b_k(\cdot)$, $b_k(y) \geq b_k(x)$ and $P_k(b_k(y)) \geq P_k(b_k(x))$. Because $b_k(y) - b_k(x)$ would get arbitrarily close to 0, so would $P_k(b_k(y)) - P_k(b_k(x))$. Suppose that we can find $k > K$ such that $b_k(y) - b_k(x) > 0$ for any K . The tie-breaking ensures that $[P_k(b_k(y)) - P_k(b_k(x))] \geq \frac{1}{N} \mu_k([x, y])$, thus, the limit of $[P_k(b_k(y)) - P_k(b_k(x))] (x - b_k(x))$ is bounded below by $\frac{1}{N} \mu([x, y]) (x - b(x))$, and this leads to a contradiction.

Claim 4. $\mu_k(I_k(\theta)) \rightarrow \mu(I(\theta))$.

Proof of Claim 4: For any $\theta \in \Theta$ such that $\theta > b(\theta)$ and $\mu(I(\theta)) > 0$, notice that

$$\limsup_k I_k(\theta) \subseteq I(\theta) \subseteq \liminf_k I_k(\theta)$$

by Claim 2 and 3. Thus, the limit of $I_k(\theta)$ exists and equals $I(\theta)$. Then,

$$\mu(I(\theta)) = \lim_k \mu(\cup_{j \geq k} I_j(\theta)) = \lim_k \mu(\cap_{j \geq k} I_j(\theta)) = \lim_k \mu(I_k(\theta)).$$

Moreover, for any $\varepsilon > 0$, there exists K' such that $\mu(\cup_{j \geq K'} I_j(\theta) - I(\theta)) < \varepsilon$. Notice that, for

any $k \geq K'$, $\mu_k(\cup_{j \geq k} I_j(\theta) - I(\theta)) \leq \mu_k(\cup_{j \geq K'} I_j(\theta) - I(\theta))$. As each $I_j(\theta)$ and $I(\theta)$ is an interval, Claim 1 implies that $\mu_k(\cup_{j \geq K'} I_j(\theta) - I(\theta))$ converges to $\mu(\cup_{j \geq K'} I_j(\theta) - I(\theta))$. Since $\lim \mu_k(\cup_{j \geq k} I_j(\theta) - I(\theta)) < \varepsilon$ an ε can be arbitrarily chosen, $\lim_{k \rightarrow \infty} \mu_k(\cup_{j \geq k} I_j(\theta) - I(\theta)) = 0$. Similarly, one can show that $\lim \mu_k(\cap_{j \geq k} I_j(\theta) - I(\theta)) = 0$, thus, $\lim \mu_k(\cap_{j \geq k} I_j(\theta)) = \lim \mu_k(\cup_{j \geq k} I_j(\theta)) = \lim_k \mu_k(I(\theta)) = \mu(I(\theta))$. Because $\mu_k(\cap_{j \geq k} I_j(\theta)) \leq \mu_k(I_k(\theta)) \leq \mu_k(\cup_{j \geq k} I_j(\theta))$, we obtain the claim $\lim_k \mu_k(I_k(\theta)) = \mu(I(\theta))$ under the assumptions above.

For $k \geq K'$ and $I(\theta)$ such that $\mu(I(\theta)) = 0$ (whether $\theta > b(\theta)$ or not), Claim 2 implies that $\limsup_k I_k(\theta) \subseteq I(\theta)$ and, thus, $\lim_k \mu_k(\cup_{j \geq k} I_j(\theta)) = 0$ and $\lim_k \mu_k(I_k(\theta)) \leq \lim_k \mu_k(\cup_{j \geq k} I_j(\theta)) = 0$. Because μ_k is a probability measure, $\lim_k \mu_k(I_k(\theta)) = 0$.

It remains to analyze the case $\mu(I(\theta)) > 0$ and there exists a type θ' in $I(\theta)$ such that $b(\theta') = \theta'$. The latter condition can only hold in equilibrium if θ' is the lowest type that has positive chance of winning in the limit. Notice that the similar proof of Claim 3 can be applied to the set $I(\theta) - \{\theta'\}$ so that $I(\theta) - \{\theta'\} \subseteq \liminf_k (I_k(\theta) - \{\theta'\})$. Therefore, if $\theta' \in \liminf_k I_k(\theta)$, thus, $I(\theta) \subseteq \liminf_k (I_k(\theta))$, then Claim 4 holds. Suppose that $\theta' \notin \liminf_k I_k(\theta)$. If there is no μ -atom at θ' then the convergence of $b_k(\theta')$ to $b(\theta')$ does not affect the winning probability of other types in $I(\theta)$ in the limit and for bidders with type θ' , it is optimal to bid $b(\theta')$ as θ' is the lowest possible bid. Suppose that $\mu(\{\theta'\}) > 0$.

Because $\theta' \notin \liminf_k I_k(\theta)$, for any K , there exists $k > K$ such that $b_k(\theta') < b_k(\theta)$. For a sufficiently large K , $b_k(\theta)$ is arbitrarily close to θ' . If θ' is an isolated point in $I(\theta)$, thus all the types above θ' in $I(\theta)$ are bounded away from θ' , the designer can increase the revenue without altering the allocation by increasing all bid levels by $\varepsilon/P_k(b_k(\theta))$ for all $\theta \neq \theta'$, for some small ε , and this leads to a contradiction. Suppose that θ' is not an isolated point in $I(\theta)$; $\theta' + \varepsilon \in I(\theta)$, for any $\varepsilon > 0$. For a large enough K , we have $b_k(x) = b_k(y)$ for all $x, y \in I(\theta) - \{\theta'\}$ and $b_k(\theta') < b_k(\theta)$. If $b_k(\theta) < \theta'$, the designer can increase the revenue similarly, and it leads to a contradiction to that $b_k(\cdot)$ is the objective maximizer. If $b_k(\theta) = \theta'$, there exists a small $\varepsilon > 0$ such that $\theta' + \varepsilon$ would deviate to $b_k(\theta')$ and it leads to a contradiction to that $b_k(\cdot)$ is in an equilibrium.

Claim 5. $\mu_k(\{x|b_k(x) < b_k(\theta)\}) \rightarrow \mu(\{x|b(x) < b(\theta)\})$.

Proof of Claim 5: Notice that if $b(x) < b(\theta)$, then there exists K such that $b_k(x) < b_k(\theta)$ for all $k > K$, thus, $\{x|b(x) < b(\theta)\} \subseteq \liminf_k \{x|b_k(x) < b_k(\theta)\}$ and $\liminf_k \mu_k(\{x|b_k(x) < b_k(\theta)\}) \geq \mu(\{x|b(x) < b(\theta)\})$.

Suppose that $\theta' \in \limsup_k \{x|b_k(x) < b_k(\theta)\}$ but $b(\theta') = b(\theta)$. By Claim 3, $\theta' \in \liminf_k I_k(\theta)$ or $\mu(I(\theta)) = 0$. If $\theta' \in \liminf_k I_k(\theta)$, then it is a contradiction to $\theta' \in \limsup_k \{x|b_k(x) < b_k(\theta)\}$, thus, $\liminf_k \mu_k(\{x|b_k(x) < b_k(\theta)\}) \geq \mu(\{x|b(x) < b(\theta)\}) \geq \limsup_k \mu_k(\{x|b_k(x) < b_k(\theta)\})$.

Suppose that $\mu(I(\theta)) = 0$. By noticing $\mu(\{x|b(x) \leq b(\theta)\}) = \mu(\{x|b(x) < b(\theta)\})$, one can easily see that $\limsup_k \mu_k(\{x|b_k(x) < b_k(\theta)\}) \leq \mu(\{x|b(x) < b(\theta)\}) \leq \liminf_k \mu_k(\{x|b_k(x) < b_k(\theta)\})$.

Claim 4 and 5 imply that $b(\cdot)$ is in an equilibrium in the limit.

It remains to show that the equilibrium $b(\cdot)$ of the first-price auction with B implements an optimal social rule. Let $M(\mu)$ denotes the set of feasible, individually rational, and incentive compatible social rules with respect to μ . For $U(y, \mu)$ and $U(y_k, \mu_k)$, define

$$f(y) = \begin{cases} U(y, \mu), & y \in M(\mu), \\ -\infty, & y \notin M(\mu). \end{cases}$$

and

$$f^k(y_k) = \begin{cases} U(y_k, \mu_k), & y_k \in M(\mu_k), \\ -\infty, & y_k \notin M(\mu_k). \end{cases}$$

For any $y_k \rightarrow y$, $\limsup_k f^k(y_k)$ is either $f(y)$ or $-\infty$, thus, $\limsup_k f^k(y_k) \leq \lim_k f^k(y_k)$. To show that $\liminf_k f^k(y_k) \geq f(y)$ for some $y_k \rightarrow y$, let us select y_k that has the reduced form of y on Θ_k . Notice that y_k is incentive compatible on Θ_k . By the continuity of U , $f^k(Q, T) = U(Q, T, \mu_k) \rightarrow U(Q, T, \mu) = f(Q, T)$ as $k \rightarrow \infty$. Therefore, f^k hypo-converges to f .

Since the social rule implemented by $b_k(\cdot)$ is optimal at each step k , any social rule implemented by a limit point of $b_k(\cdot)$ is optimal by Theorem 7.33 in Rockafellar and Wets (1998). Because we already established that b is a limit point of b_k , it is optimal.

Proof of Lemma 3

Proof. (i) \iff (iii): For this proof, let us introduce a notation for CDF of π , $F(\theta) = \sum_{x < \theta} \pi(x)$. Notice that the set \mathcal{Q} in (6) is characterized by $2T$ linear constraints as follows.

$$\mathcal{Q} = \{Q : \Theta \rightarrow [0, 1] \mid N \sum_{s=t}^T Q(\theta^s) \pi(\theta^s) \leq 1 - F(\theta^t)^N, Q(\theta^t) \geq Q(\theta^{t-1}),$$

for all $t = 1, \dots, T\}$.

Let us denote $N \sum_{s=t}^T Q(\theta^s) \pi(\theta^s) \leq 1 - F(\theta^t)^N$ by “ t -th feasibility constraint”. Denote $Q(\theta^t) \geq Q(\theta^{t-1})$ by “ t -th monotonicity constraint” for $t > 1$. First of all, notice that any T constraints in \mathcal{Q} are linearly independent. Suppose that Q is an extreme point of \mathcal{Q} and denote the partition that characterizes Q by $\{\Theta_1, \dots, \Theta_J\}$. $T - J$ number of monotonicity constraints are binding. First, we show that there exists $\theta^{t_j} \in \Theta_j$ such that t_j -th feasibility constraint is binding, for each $j > 1$. Suppose not. Then there exists Θ_j such that no

feasibility constraints are binding for all $\theta \in \Theta_j$. Define Q' and Q'' as follows.

$$Q'(\theta) = \begin{cases} Q(\theta) & \text{if } \theta \notin \Theta_j, \\ Q(\theta) + \epsilon & \text{if } \theta \in \Theta_j. \end{cases}$$

$$Q''(\theta) = \begin{cases} Q(\theta) & \text{if } \theta \notin \Theta_j, \\ Q(\theta) - \epsilon & \text{if } \theta \in \Theta_j. \end{cases}$$

By taking ϵ small enough, both Q' and Q'' are in \mathcal{Q} . Moreover, $0.5Q' + 0.5Q'' = Q$. This is contradiction to that Q is an extreme point. Notice that if $Q(\theta_1)$ is 0, Q'' is not well-defined, thus, the feasibility constraint does not need to be binding for $j = 1$. However, if $Q(\Theta_1) > 0$, the feasibility constraint for $j = 1$ must be binding by the same argument. As a next step, we show that $\theta^{t_j} = \min \Theta_j$ for each $j > 1$. We will prove this by induction. Before proceeding further with the proof, the following variation of geometric summation formula will be useful in reading the proof.

$$\frac{x^n - y^n}{x - y} = \sum_{s=0}^{n-1} x^{n-1-s} y^s.$$

Consider Θ_J and θ^{t_J} . Since t_J -th feasibility constraint is binding,

$$NQ(\Theta_J)(1 - F(\theta^{t_J})) = 1 - F(\theta^{t_J})^N,$$

and

$$NQ(\Theta_J) = \frac{1 - F(\theta^{t_J})^N}{1 - F(\theta^{t_J})} = \sum_{s=0}^{N-1} F(\theta^{t_J})^s.$$

Suppose that there exists $\theta \in \Theta_J$ such that $\theta < \theta^{t_J}$. The feasibility constraint evaluated at θ is

$$NQ(\Theta_J)(1 - F(\theta)) \leq 1 - F(\theta)^N,$$

and

$$NQ(\Theta_J) \leq \frac{1 - F(\theta)^N}{1 - F(\theta)} = \sum_{s=0}^{N-1} F(\theta)^s. \quad (7)$$

This is a contradiction because $F(\theta^{t_j}) > F(\theta)$. Now, suppose that $\theta^{t_{j+1}} = \min \Theta_{j+1}$ and consider Θ_j and θ^{t_j} . If $\theta^{t_j} > \min \Theta_j$, there exist $\theta \in \Theta_j$ such that $\theta < \theta^{t_j}$. Since t_j -th and t_{j+1} -th feasibility constraints are binding,

$$\begin{aligned} NQ(\Theta_j)(F(\theta^{t_{j+1}}) - F(\theta^{t_j})) + N \sum_{x \geq \theta^{t_{j+1}}} Q(x)\pi(x) &= 1 - F(\theta^{t_j})^N, \\ N \sum_{x \geq \theta^{t_{j+1}}} Q(x)\pi(x) &= 1 - F(\theta^{t_{j+1}})^N. \end{aligned}$$

By taking the difference,

$$NQ(\Theta_j) = \frac{F(\theta^{t_{j+1}})^N - F(\theta^{t_j})^N}{F(\theta^{t_{j+1}}) - F(\theta^{t_j})} = \sum_{s=0}^{N-1} F(\theta^{t_{j+1}})^{N-1-s} F(\theta^{t_j})^s.$$

By evaluating the feasibility constraint at θ , the same derivation leads to

$$NQ(\Theta_j) \leq \sum_{s=0}^{N-1} F(\theta^{t_{j+1}})^{N-1-s} F(\theta)^s,$$

which is a contradiction since $F(\theta^{t_j}) > F(\theta)$. The above step also shows that

$$\begin{aligned} NQ(\Theta_j) &= \frac{F(\theta^{t_{j+1}})^N - F(\theta^{t_j})^N}{F(\theta^{t_{j+1}}) - F(\theta^{t_j})} = \frac{F(\theta^{t_{j+1}})^N - F(\theta^{t_j})^N}{\pi(\Theta_j)}, \\ &= \frac{(\sum_{k=1}^j \pi(\Theta_k))^N - (\sum_{k=1}^{j-1} \pi(\Theta_k))^N}{\pi(\Theta_j)}, \text{ for } j > 1. \end{aligned}$$

For $j = 1$, if $Q(\Theta_1) > 0$, then the binding feasibility constraint leads to

$$NQ(\Theta_1) = \frac{\pi(\Theta_1)^N}{\pi(\Theta_1)} = \pi(\Theta_1)^{N-1}.$$

For $j = 1$, if $Q(\theta_1) = 0$, then $Q(\theta_1) \geq 0$ constraint must be binding instead of the feasibility constraint since we need T number of binding constraints for Q to be an extreme point. For converse, if Q takes the form in (iii), it is easy to see that T number of constraints are binding at Q .

(ii) \iff (iii):

$$\begin{aligned} Q(\Theta_j) &= \sum_{n=1}^N \frac{1}{n} \binom{N-1}{n-1} \pi(\Theta_j)^{n-1} \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n} \\ &= \frac{1}{N} \sum_{n=1}^N \binom{N}{n} \pi(\Theta_j)^{n-1} \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n}, \end{aligned}$$

because $\frac{N}{n} \binom{N-1}{n-1} = \binom{N}{n}$. Applying the binomial formula,

$$\begin{aligned} & \frac{1}{N} \sum_{n=1}^N \binom{N}{n} \pi(\Theta_j)^{n-1} \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n} \\ &= \frac{1}{N\pi(\Theta_j)} \sum_{n=1}^N \binom{N}{n} \pi(\Theta_j)^n \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n} \\ &= \frac{1}{N\pi(\Theta_j)} \left[\sum_{n=0}^N \binom{N}{n} \pi(\Theta_j)^n \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^{N-n} - \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^N \right] \end{aligned} \quad (8)$$

$$= \frac{1}{N\pi(\Theta_j)} \left[\left[\sum_{l=1}^{j-1} \pi(\Theta_l) + \pi(\theta_j) \right]^N - \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^N \right] \quad (9)$$

$$= \frac{1}{N\pi(\Theta_j)} \left[\left[\sum_{l=1}^j \pi(\Theta_l) \right]^N - \left[\sum_{l=1}^{j-1} \pi(\Theta_l) \right]^N \right]. \quad (10)$$

The binomial formula is used from (8) to (9). If $j = 1$, (10) becomes $\frac{\pi(\Theta_1)^{N-1}}{N}$. If the seller keeps the item when the all bidders' types are in Θ_1 , then $Q(\Theta_1)$ is 0. To show the converse, we established that if Q satisfies (49) and (50), then $Q(\Theta_{j+1}) > Q(\Theta_j)$ because

$$NQ(\Theta_j) = \frac{F(\theta^{t_{j+1}})^N - F(\theta^{t_j})^N}{F(\theta^{t_{j+1}}) - F(\theta^{t_j})} = \sum_{s=0}^{N-1} F(\theta^{t_{j+1}})^{N-1-s} F(\theta^{t_j})^s$$

Thus, the hierarchical allocation rule with a partition $\{\Theta_1, \dots, \Theta_J\}$ implements Q . \square

Proof of Lemma 4

Proof. Denote the coefficient of Q_i^K by α_k . It is enough to show that the following system is feasible, where $T_i^k(\theta^t)$'s are variables.

$$\begin{aligned} Q_i^k(\theta^t)\theta^t - T_i^k(\theta^t) &\geq Q_i^k(\theta^{t-1})\theta^t - T_i^k(\theta^{t-1}), \text{ for } t = 2, \dots, T \text{ and all } k, \\ Q_i^k(\theta^{t-1})\theta^{t-1} - T_i^k(\theta^{t-1}) &\geq Q_i^k(\theta^t)\theta^{t-1} - T_i^k(\theta^t), \text{ for } t = 2, \dots, T \text{ and all } k, \\ Q_i^k(\theta^1)\theta^1 - T_i^k(\theta^1) &\geq 0 \text{ for all } k, \\ \sum_{k=1}^K \alpha_k T_i^k(\theta^t) &= T_i(\theta^t), \text{ for all } t = 1, \dots, T. \end{aligned}$$

Let $\Delta Q_i^k(\theta^t) = Q_i^k(\theta^t) - Q_i^k(\theta^{t-1})$, then the system of inequalities are

$$\begin{aligned} T_i^k(\theta^t) - T_i^k(\theta^{t-1}) &\leq \Delta Q_i^k(\theta^t)\theta^t, \text{ for } t = 1, \dots, T \text{ and all } k, \\ -T_i^k(\theta^t) + T_i^k(\theta^{t-1}) &\leq -\Delta Q_i^k(\theta^t)\theta^{t-1}, \text{ for } t = 2, \dots, T \text{ and all } k, \\ \sum_{k=1}^K \alpha_k T_i^k(\theta^t) &= T_i(\theta^t), \text{ for all } t = 1, \dots, T. \end{aligned}$$

Denote the dual variables corresponding to $\Delta Q_i^k(\theta^t)\theta^s$ by $z_{t,s}^k$ and the dual variables corresponding to $T_i(\theta^t)$ by y_t . Then, the Farkas alternative of the original system is

$$-z_{t+1,t+1}^k + z_{t+1,t}^k + z_{t,t}^k - z_{t,t-1}^k + \alpha_k y_t = 0, \text{ for all } t \text{ and } k, \quad (11)$$

$$\sum_{t=1}^T \sum_{k=1}^K [z_{t,t}^k \Delta Q_i^k(\theta^t)\theta_t - z_{t,t-1}^k \Delta Q_i^k(\theta^t)\theta^{t-1}] + \sum_{t=1}^T y_t T_i(\theta^t) < 0. \quad (12)$$

By rearranging (11) and (12),

$$z_{t,t}^k - z_{t,t-1}^k = -\alpha_k \sum_{s=t}^T y_s, \quad (13)$$

$$\sum_{t=1}^T \sum_{k=1}^K [(z_{t,t}^k - z_{t,t-1}^k) \Delta Q_i^k(\theta^t)\theta^t + \Delta Q_i^k(\theta^t)(\theta^t - \theta^{t-1})z_{t,t-1}^k] + \sum_{t=1}^T y_t T_i(\theta^t) < 0. \quad (14)$$

Observe that

$$\sum_{t=1}^T y_t T_i(\theta^t) = \sum_{t=1}^T \sum_{s=t}^T y_s [(T(\theta^t) - T(\theta^{t-1}))]. \quad (15)$$

Now, assume that the Farkas alternative is feasible. By substituting (13) and (15) into (14),

$$\begin{aligned}
& \sum_{t=1}^T \sum_{k=1}^K [-\alpha_k \sum_{s=t}^T y_s \Delta Q_i^k(\theta^t) \theta^t + \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k] + \sum_{t=1}^T y_t T_i(\theta^t) \\
&= \sum_{t=1}^T \sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k - \sum_{t=1}^T (\sum_{s=t}^T y_s) \Delta Q_i(\theta^t) \theta^t + \sum_{t=1}^T (\sum_{s=t}^T y_s) (T_i(\theta^t) - T_i(\theta^{t-1})) \\
&= \sum_{t=1}^T \left[\sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k + (\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^t] \right], \tag{16}
\end{aligned}$$

$$\geq \sum_{t=1}^T \left[\sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) \alpha^k \sum_{s=t}^T y_s + (\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^t] \right], \tag{17}$$

$$= \sum_{t=1}^T (\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^{t-1}]$$

where

$$\Delta Q_i(\theta^t) = Q_i(\theta^t) - Q_i(\theta^{t-1}).$$

The inequality between (16) and (17) is from (13). Notice that

$$\sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k \geq 0, \tag{18}$$

$$T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^t \leq 0, \tag{19}$$

$$T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^{t-1} \geq 0. \tag{20}$$

If $\sum_{s=t}^T y_s \leq 0$, then for such t ,

$$\sum_{k=1}^K \Delta Q_i^k(\theta^t) (\theta^t - \theta^{t-1}) z_{t,t-1}^k + (\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^t] \geq 0.$$

If $\sum_{s=t}^T y_s > 0$, then for such t ,

$$(\sum_{s=t}^T y_s) [T_i(\theta^t) - T_i(\theta^{t-1}) - \Delta Q_i(\theta^t) \theta^{t-1}] \geq 0.$$

Therefore the Farkas' alternative is infeasible and the original system is feasible. \square

Proof of Theorem 5

Proof. Denote $b_\theta = T(\theta)/Q(\theta)$ for each θ such that $Q(\theta) > 0$, where Q is the reduced form of $\{q_i\}_{i=1}^N$. We will show that $b_\theta > b_{\theta'}$ if and only if $Q(\theta) > Q(\theta')$. Consider θ^t such that $Q(\theta^t) > 0$. Since (Q, T) is incentive compatible,

$$\begin{aligned} Q(\theta^t)\theta^t - T(\theta^t) &\geq Q(\theta^{t+1})\theta^t - T(\theta^{t+1}) \\ \iff \frac{Q(\theta^t)}{Q(\theta^{t+1})} \left[\theta^t - \frac{T(\theta^t)}{Q(\theta^t)} \right] &\geq \theta^t - \frac{T(\theta^{t+1})}{Q(\theta^{t+1})} \end{aligned}$$

If $Q(\theta^t) < Q(\theta^{t+1})$, $b_{\theta^t} < b_{\theta^{t+1}}$ by the inequality. If $Q(\theta^t) = Q(\theta^{t+1})$, then $T(\theta^t) = T(\theta^{t+1})$ by the incentive compatibility. Since b_θ increases if and only if $Q(\theta)$ increases, if each bidder with type θ places a bid b_θ , it will implement the monotonic hierarchical $\{q_i\}_{i=1}^N$. The expected payment when bidding b_θ is $Q(\theta^t)b_\theta = T(\theta)$ by construction. Thus, it is an equilibrium strategy for a bidder with type θ to place a bid b_θ if $Q(\theta) > 0$. If $Q(\theta) = 0$, bidders with type θ choose to opt out in the equilibrium. \square