Double Auctions: A Unified Treatment[*]

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Abstract

We provide the first definition of a Double Auction that applies to both finite and infinite markets and does not rely on any regularity assumptions. In particular, our definition allows ties in reported values. In all markets, our Double Auction clears the market, and implements a Walrasian equilibrium. Our Double Auction nests as a special cases the standard k-Double Auction for finite markets and the Double Auction for continuous and strictly monotone demand and supply for infinite markets. We also show that the convergence of finite to infinite Double Auctions obtains in absence of any regularity assumptions.

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1 Introduction

Double auctions are amongst the most ubiquitous market mechanisms that clear the market, matching demand and supply. They are at the core of stock, asset, and commodity markets on many platforms and exchanges. Potential buyers and sellers submit buy and sell orders to a central clearinghouse that then establishes who is involved in trade and what the market price is. The market price is selected to clear the market, that is, to equilibrate demand and supply. The idea to match demand and supply has been a central in economic theory, going back to Cournot, Jenkin, Marshall, Walras and other forefathers of economics.[1]

We provide a unified double auction mechanism that applies to both finite and infinite markets and does not impose any regularity assumptions on demand and supply. For markets with finitely many traders, double auction mechanism were formally defined as the k-Double Auction (Wilson, 1985; Rustichini et al., 1994),[2] who provide explicit formulae for calculating market clearing prices that do not extend to infinite markets. For markets with infinitely many traders, the definition of double auction mechanisms rely on regularity assumptions on the elasticity of demand and supply such as continuity and strict monotonicity, cf. Mas-Colell et al. (1995) and Reny and Perry (2006); these assumptions ensure that a market clearing price exists and is unique. Our unified framework for finite and infinite markets allows to carry over results from the more tractable infinite model to approximate large finite markets.

The regularity assumptions of the prior literature on large double auctions are satisfied in certain settings but fail in other settings of economic importance. For instance, they fail if the demand is locally perfectly elastic or locally perfectly inelastic.[3] The key reason why demand (or supply) is perfectly elastic are ties between bids (and resulting atoms in the distribution of bids) while inelasticity is created by the absence of certain bids. Ties and gaps—and hence the failure of regularity—may occur for various reasons. They are caused by the indivisibility

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[1] See, for example, Friedman and Rust (1993) for an overview of the use of double auctions in markets, and Humphrey (1992) for a history of the concept.

[2] The two formulations are equivalent; we shall use the formulation by Rustichini et al. (1994). See also earlier formulations for the bilateral k-double auction case by Chatterjee and Samuelson (1983) and Satterthwaite and Williams (1989a).

[3] Perfect elasticity creates discontinuities of demand, which may break the existence of a market clearing price as recognized by Rustichini et al. (1994). Perfect inelasticity makes demand constant in some price regions, and there might exist multiple market clearing prices.

of currency: \$0.01 is often the smallest bid increment in stock trading, \$0.0001 in currency trading, while other markets have much larger increments, e.g., art auctions and NFT trading.[4] The presence of large buy or sell orders leads to ties. Ties might also result from traders strategically bidding the same as they aim to influence the market price or bid-related fee payments (Jackson et al., 2002; Woodward, 2019; Jantschgi et al., 2022); strategic bidding might also lead to traders avoiding certain bids. Finally, ties arise when traders with common values herd (Banerjee, 1992; Bikhchandani et al., 1992) or when traders are boundedly rational and imitate each other (Shiller, 2000).

The main contributions of this article are as follows. We define the unified Double Auction mechanism. Our Theorem 1 shows that it implements market clearing and Walrasian equilibrium with regard to revealed demand and supply.[5] Theorem 2 shows that our Double Auction nests earlier formulations including the k-Double Auction in finite markets and continuous and strictly monotone demand and supply models in infinite markets. Theorems 3 and 4 establish the convergence of finite to infinite markets. The table, presented at the end of this section, summarizes the prior state of knowledge alongside the contributions that our unified framework has introduced.

While we are not concerned with strategic considerations, there is a large body of work that shows that the double auction is asymptotically truthful and efficient, thus providing a theoretical justification for its widespread use in practice, see. e.g., Wilson (1985), Rustichini et al. (1994), Satterthwaite and Williams (2002), Jackson and Swinkels (2005), Tatur (2005), Cripps and Swinkels (2006), Reny and Perry (2006), and Azevedo and Budish (2019). These papers rely on prior definitions of the double auction mechanism and the aforementioned regularity assumptions. Jantschgi et al. (2022) relies on our definition to study how transaction costs affect the strategic behavior in double auctions; as discussed above, the presence of transaction costs leads to the violation of the earlier regularity assumptions.

Our paper belongs to the recent wave of work extending canonical mecha-

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[4]See U.S. Securities and Exchange Commission (2005) for stock trading, Bank for International Settlements (2019) for the foreign exchange market, and OpenSea (2022)'s User Guide for a large NFT trading platform.

[5]A Walrasian equilibrium consists of an allocation (including prices), such that no trader – given their revealed preferences – desires any other affordable bundle. Prior to our work, the link between double auctions and Walrasian equilibrium had only be shown under the above mentioned regularity assumptions; see Wilson (1985); Rustichini et al. (1994).

nisms from finite to infinite markets and showing convergence between the two: Abdulkadiroğlu et al. (2015) and Azevedo and Leshno (2016) extended deferred acceptance, Che and Kojima (2010) extended probabilistic serial and random serial dictatorship, and Leshno and Lo (2020) extended top trading cycle mechanisms.

	Finite markets		Convergence	Infinite markets	
	Mechanism	*Walrasian link*	$\longrightarrow$	*Mechanism*	*Walrasian link*
With regularity assumptions[a]	W85, RSW94	W85[b], RSW94[b]	RP06	MWG95, RP06	MWG95, RP06[b]
W/o regularity assumptions	W85, RSW94	**Proposition 1**	**Theorem 3 & 4**	**Theorem 1**	**Proposition 1**

Summary of prior state of knowledge and our contribution

Unified framework (Theorem 1 & 2)

Abbreviations: Wilson (1985), Rustichini, Satterthwaite, and Williams (1994), Mas-Colell, Whinston, and Green (1995), Reny and Perry (2006)

[a]For finite markets, this implies no ties. In infinite markets, Mas-Colell et al. (1995) examines the standard model of continuous, strictly monotone demand and supply, while Reny and Perry (2006) addresses equilibrium-driven supply and demand with added assumptions on analytical properties and discrete bids and asks.
[b]These papers show that there exists a market clearing price equating demand and supply, which essentially provides a link to Walrasian equilibrium, see Proposition 1 below.

2   The market

We study two-sided markets with traders interested in either buying or selling a finite number of indivisible units of a good. Each trader can submit multiple single-unit buy or sell orders. We study double auctions, market clearing, and Walrasian equilibria taking these orders as primitives.

Denote by $\mathcal{B} \subset \mathbb{R}$ the set of buy orders ($b \in \mathcal{B}$) and by $\mathcal{S} \subset \mathbb{R}$ the set of sell orders ($s \in \mathcal{S}$). Consider two cases: a market with finite numbers of orders on each market side (*finite* market) and a market with continuum of orders on each market side (*infinite* market). When the market has m buy orders and n sell orders, label orders as $\mathcal{B} = \{1, 2, ..., m\}$ and $\mathcal{S} = \{1, 2, ..., n\}$. When the market has a continuum of orders, the labels are real numbers from closed intervals $\mathcal{B} \subset \mathbb{R}$ and $\mathcal{S} \subset \mathbb{R}$. Denote by μ_B and μ_S the counting measure (in the finite case) or

the Lebesgue measure (in the infinite case) on the sets $\mathcal{B}$ and $\mathcal{S}$.

Every order i includes a *value* $v_i \in V = [\underline{v}, \overline{v}] \subset \mathbb{R}^+$, where V is the *space* of possible values. We assume that the functions $v_B : \mathcal{B} \to V$ and $v_S : \mathcal{S} \to V$ that assign values to orders are Borel. A buy order's value specifies a *bid*, that is, the maximum willingness to pay. A sell order's value specifies an *ask*, that is, the minimum willingness to sell.

Given all buy and sell orders, a double auction chooses a market outcome determined by a market price P^* and an allocation identifying subsets of orders, $\mathcal{B}^* \subset \mathcal{B}$ and $\mathcal{S}^* \subset \mathcal{S}$, that are filled. For every filled buy order the associated trader buys one unit of the good, and for every filled sell order the associated trader sells one unit of the good. The market price is what each trader pays or receives for every filled order.

What remains to be done is to define the market price and the subsets of filled orders $\mathcal{B}^*$ and $\mathcal{S}^*$. As discussed in Section 1, the preexisting definitions relied on market finiteness or ad hoc regularity assumptions. We address this question in general, starting with the auxiliary concepts of market clearing and Walrasian equilibria.

3   Market clearing and Walrasian equilibria

The market price in a double auction should be market clearing, that is balance supply and demand. The terms market clearing price and Walrasian equilibrium price are often used interchangeably, albeit having different definitions. We next provide formal definitions of market clearing and Walrasian equilibrium via demand and supply, and show their equivalence.

A series of auxiliary lemmas on analytical properties of demand and supply, as well as the geometrical properties of the set of market clearing prices in our framework, is relegated to Appendix A. The lemmas are instructive to understand our main results but they are known for other models (e.g., finite economies or economies with divisible goods).[6]

Consider the set of all orders with reported bid or ask (strictly) above or

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[6] For perfectly divisible goods, see, for example, Debreu (1959) and Arrow and Hahn (1971) for markets with finitely many orders and Hildenbrand (1974) for markets with a continuum of orders. For a discussion of demand and supply in the finite k-Double Auction, see, for example, Rustichini et al. (1994). For our most general case, a market with indivisible goods and a continuum of orders, see, for example, Mas-Colell (1977) and Azevedo et al. (2013) for some analytical properties of demand and supply correspondences.

below a price P. For a relation $\mathcal{R} \in \{\geq, >, =, <, \leq\}$, we introduce the shorthand notation $\mathcal{B}_{\mathcal{R}}(P) = \{b \in \mathcal{B} : v_b \mathcal{R} P\}$ and $\mathcal{S}_{\mathcal{R}}(P) = \{s \in \mathcal{S} : v_s \mathcal{R} P\}$.

Demand at price P is defined as the mass of all buy orders with bid above P and, similarly, supply at price P is defined as the mass of all sell orders with ask below P. To account for orders with a bid or ask equal to a price P, we distinguish between weak and strict inequalities by defining demand and supply functions and correspondences.

Definition (Demand and supply). The *demand* and *supply functions* at price P are $D_f(P) = \mu_B(\mathcal{B}_{\geq}(P))$ and $S_f(P) = \mu_S(\mathcal{S}_{\leq}(P))$. The *demand* and *supply correspondences* are the set-valued functions $D_c(P) = [\mu_B(\mathcal{B}_{>}(P)), \mu_B(\mathcal{B}_{\geq}(P))]$ and $S_c(P) = [\mu_S(\mathcal{S}_{<}(P)), \mu_S(\mathcal{S}_{\leq}(P))]$.

Analytical properties of demand and supply functions, as well as the connection to their correspondences are relegated to Appendix A.

Other market metrics we will use are how much trade is possible at price P, and how big the difference is between demand and supply at that price.

Definition (Trade volume and excess). The maximally achievable *trade volume* at price P is $Q(P) = \min(D_f(P), S_f(P))$ and the minimally achievable excess $Ex(P) = |D_f(P) - S_f(P)|$. If $D_f(P) > S_f(P)$, call $Ex(P)$ *excess demand*, and if $D_f(P) < S_f(P)$, call $Ex(P)$ *excess supply*.

The trade volume $Q(P)$ at price P is the maximal possible mass of trade that can be facilitated at price P, while respecting individual rationality.

We now define the set of prices that balance demand and supply, that is, prices, where the demand and supply correspondences intersect.

Definition (Market clearing prices). A price P is a *market clearing price*, if $D_c(P) \cap S_c(P) \neq \emptyset$. Denote the set of all market clearing prices by $\mathcal{P}_{MC}$.

We discuss the connection between market clearing, trade volume, and excess in Appendix A. In Lemma 5 we observe that the set of market clearing prices is non-empty, convex, and closed. However, the excess at a market clearing price is not necessarily equal to zero.

We define Walrasian equilibrium with respect to submitted buy and sell orders. In equilibrium, the market clears, that is, the same number of buy and sell orders are filled (*trade-balance*). Every buy order with bid strictly above the market price

and every sell order with ask strictly below the market price are filled (*stability*). No buy order is filled at a price above its bid price and no sell order is filled at a price below its ask price (*individual rationality*). Formally:

Definition (Walrasian equilibrium)**.** A market outcome $(P^*, \mathcal{B}^*, \mathcal{S}^*)$ is a *Walrasian equilibrium*, if it *balances trade* ($\mu_B(\mathcal{B}^*) = \mu_S(\mathcal{S}^*)$), it is *stable* ($\mathcal{B}_>(P^*) \subset \mathcal{B}^*$ and $\mathcal{S}_<(P^*) \subset \mathcal{S}^*$), and it is *individually rational* ($\mathcal{B}^* \subset \mathcal{B}_\geq(P^*)$ and $\mathcal{S}^* \subset \mathcal{S}_\leq(P^*)$).

The sets of market clearing prices and Walrasian equilibrium prices coincide.

Proposition 1 ($\mathcal{P}_{MC} = \mathcal{P}_{EQ}$)**.** *A price is a market clearing price if and only if it is a Walrasian equilibrium price.*

We provide the proof in Appendix B.1.[7]

4 Main results

In this section we define the Double Auction and show that it results in market clearing and Walrasian equilibrium. Moreover, we show that it nests the k-Double Auction in finite markets and continuous and strictly monotone demand and supply models as special cases, and establish convergence of finite to infinite markets.

4.1 The Double Auction

We here present our unified mechanism of double auctions for finite and infinite markets. Recall that at the end of Section 2, we defined such auctions leaving open the question how to determine the market price and the sets of filled orders.

Definition (Double Auction: Market Price and Filled Orders)**.** Given all buy and sell orders, the Double Auction sets a market price P^* in the interval of market clearing prices

$$P^* \in [\min \mathcal{P}_{MC}, \ \max \mathcal{P}_{MC}]$$

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[7]The existence of Walrasian equilibria in markets with unit demand for an indivisible good has been established, see, e.g., Quinzii (1984) for finite markets and Mas-Colell (1977) for infinite markets. Within our framework, this follows from Proposition 1 and the existence of market clearing prices (Lemma 5) in Appendix A.

and fills the following sets of buy orders $\mathcal{B}^*$ and sell orders $\mathcal{S}^*$:

$$\mathcal{B}^* = \mathcal{B}_{\geq}(P^*) \qquad \text{and } \mathcal{S}^* = \mathcal{S}_{\leq}(P^*) \qquad \text{if there is no excess at } P^*$$

$$\mathcal{B}^* = \mathcal{B}_{>}(P^*) \cup \tilde{\mathcal{B}} \text{ and } \mathcal{S}^* = \mathcal{S}_{\leq}(P^*) \qquad \text{if there is excess demand at } P^*$$

$$\mathcal{B}^* = \mathcal{B}_{\geq}(P^*) \qquad \text{and } \mathcal{S}^* = \mathcal{S}_{<}(P^*) \cup \tilde{\mathcal{S}} \quad \text{if there is excess supply at } P^*$$

$\tilde{\mathcal{B}} \subset \mathcal{B}_{=}(P^*)$ and $\tilde{\mathcal{S}} \subset \mathcal{S}_{=}(P^*)$ are sets of orders selected according to some rationing rule to ensure that the trade is balanced, that is $\mu_B(\mathcal{B}^*) = \mu_S(\mathcal{S}^*)$.

The Double Auction allows for arbitrary pricing and rationing rules. Examples of pricing rules include setting the market price P^* as a convex combination of the endpoints of $\mathcal{P}_{MC}$ (Rustichini et al., 1994) or choosing the price uniformly at random. A standard example of a rationing rule is to select orders uniformly at random.[8] We prove the existence of a uniform rationing rule in Lemma 11. Other pricing and rationing rules make the choice endogenous based on aspects of the environment that are not part of our model, e.g., based on the timing with which buy or sell orders were submitted or based on the volume of the total order.

Theorem 1 (Properties of the mechanism)**.** *The Double Auction is well-defined for any market instance, it maximizes the trade volume, and implements market clearing and Walrasian equilibrium.*

We provide the proof in Appendix B.2.

4.2   Nesting of existing mechanisms

For the case of finitely many orders, the standard double auction mechanism is Rustichini et al. (1994)'s formulation of the k-Double Auction, which extends the bilateral models of Chatterjee and Samuelson (1983) and Satterthwaite and Williams (1989a) and is equivalent to an earlier formulation of Wilson (1985).

Definition (Finite k-Double Auction (Rustichini et al., 1994))**.** Given all $m + n$ bids and asks associated to m buy orders and n sell orders, denoted by the set v, define by $v^{(l)}$ its l'th smallest element. For $k \in [0, 1]$, the k-Double Auction sets

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[8]Buy orders in $\tilde{\mathcal{B}}$ are selected uniformly at random from $\mathcal{B}_{=}(P^*)$ if the probability $\mathbb{P}[b \in \tilde{\mathcal{B}}]$ is the same for all $b \in \mathcal{B}_{=}(P^*)$. The random selection from $\mathcal{S}_{=}(P^*)$ is defined analogously.

the market price P^* as

$$P^* = kv^{(m)} + (1 - k)v^{(m+1)},$$

and fills all buy orders with bid strictly above and all sell orders with ask strictly below the market price to be involved in trade. For all $P^* \in (v^{(m)}, v^{(m+1)})$, the numbers of such buy and sell orders are equal and the mechanism terminates.[9] Furthermore, when $P^* = v^{(m)}$ or $P^* = v^{(m+1)}$, the maximum number of buy or sell orders with bid or ask equal to the market price is included in trade such that the final numbers of filled orders on both market sides are equal. In case there are more orders with bid or ask equal to the market price than can be selected, a fair lottery selects the filled orders.

In infinite markets, double auction mechanisms have been defined for textbook models with continuous and strictly monotone demand and supply functions with a unique intersection P^*, see, e.g., Mas-Colell et al. (1995) and Reny and Perry (2006).[10]

Definition (Continuous and strictly monotone demand and supply models (Reny and Perry, 2006; Mas-Colell et al., 1995))**.** The market price P^* is set as the unique intersection of the continuous and strictly monotone demand and supply functions, that is $D_f(P^*) = S_f(P^*)$. The sets of filled buy and sell orders are $\mathcal{B}^* = \mathcal{B}_{\geq}(P^*)$ and $\mathcal{S}^* = \mathcal{S}_{\leq}(P^*)$ (no tie-breaking).

The Double Auction nests the k-Double Auction for finite markets and continuous and strictly monotone demand and supply models for infinite markets as special cases.

Theorem 2 (Nesting of prior mechanisms)**.** *The Double Auction nests as special cases the k-Double Auction for finite markets (Wilson, 1985; Rustichini et al., 1994) and continuous and strictly monotone demand and supply models (Mas-Colell et al., 1995; Reny and Perry, 2006).*

We provide the proof in Appendix B.3.

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[9]We provide a detailed discussion in Lemma 7 in Appendix A.

[10]In the presence of ties due to a discretization of the action space, Reny and Perry (2006, Online Supplement) provide an argument for tie-breaking in infinite markets. It can be checked straightforwardly that the tie-breaking rule is equivalent to the allocation rule of the Double Auction. However, we omit a proof, because they do not provide a full mechanism in the presence of ties.

4.3 Convergence

We establish convergence properties of the set of market clearing prices from finite to infinite markets. For $k = 1, 2, 3, ...$, a sequence of finite markets $(\mathcal{B}^k, \mathcal{S}^k, v_B^k, v_S^k)$ is a *growing sequence of markets*, if for all $k \geq 1$ it holds that $\mathcal{B}^k \subset \mathcal{B}^{k+1}$, $\mathcal{S}^k \subset \mathcal{S}^{k+1}$, and for all $b \in \mathcal{B}^k$ and $s \in \mathcal{S}^k$ it holds that $v_B^k(b) = v_B^{k+1}(b)$ and $v_S^k(s) = v_S^{k+1}(s)$. We normalize measures μ_B^k and μ_S^k, and therefore demand and supply, by multiplying with factors $\frac{\beta}{|\mathcal{B}^k|}$ and $\frac{\sigma}{|\mathcal{S}^k|}$, with β, σ strictly positive and finite real numbers. Therefore, the measures μ_B^k and μ_S^k in the sequence of finite markets have fixed mass β and σ, corresponding to the proportion of buy and sell orders. A growing sequence of finite markets is *convergent* with *limit* $(\mathcal{B}, \mathcal{S}, v_B, v_S)$ if for all $k \geq 1$ it holds that $\mathcal{B}^k \subset \mathcal{B}$, $\mathcal{S}^k \subset \mathcal{S}$, $v_B^k(b) = v_B(b)$ and $v_S^k(s) = v_S(s)$, and $\lim_{k \to \infty} \sup_{P \in \mathbb{R}} |D_f^k(P) - D_f(P)| = 0$ and $\lim_{k \to \infty} \sup_{P \in \mathbb{R}} |S_f^k(P) - S_f(P)| = 0$. That is, the demand and supply functions converge uniformly. Denote by $\mathcal{P}_{MC}^k$ and $\mathcal{P}_{MC}$ the corresponding sets of market clearing prices in the sequence of finite markets and in the limit market.

For two non-empty sets $A, B \subset \mathbb{R}$ with representative points $a \in A$ and $b \in B$, we define the *distance* between a and b as the Euclidean distance $d(a, b) = |a - b|$, between a and B as $d(a, B) = \inf_{b \in B} d(a, b)$, and between A and B as $d(A, B) = \inf_{a \in A} d(a, B) = \inf_{b \in B} d(A, b)$. Denote the *diameter* of a set A as $diam(A) = |\sup A - \inf A|$.

Theorem 3 (Convergence). *For any convergent sequence of finite markets* $(\mathcal{B}^k, \mathcal{S}^k, v_B^k, v_S^k)$ *with limit* $(\mathcal{B}, \mathcal{S}, v_B, v_S)$, *it holds that*

$$\lim_{k \to \infty} d(\min \mathcal{P}_{MC}^k, \mathcal{P}_{MC}) = 0 \quad \text{and} \quad \lim_{k \to \infty} d(\max \mathcal{P}_{MC}^k, \mathcal{P}_{MC}) = 0.$$

We provide the proof in Appendix B.4. Note that by Proposition 1 the sequence of Walrasian equilibrium prices has the same convergence property. If a unique market clearing price exists in the limit, then Theorem 3 implies that $\lim_{k \to \infty} diam(\mathcal{P}_{MC}^k) = 0$, i.e., the set of market clearing prices becomes small in large markets.

In economic analyses of finite markets it is common to assume that agents' preferences are drawn from continuous distributions (e.g., Myerson and Satterthwaite 1983; Chatterjee and Samuelson 1983; Rustichini et al. 1994). Are such draws likely to generate convergent sequences of economies as we grow the market size?

We answer this question in the positive, thus closing the loop on the relationship between finite and infinite markets.

We study randomly generated economies and show that as the market size grows they convergent almost surely to a limit infinite market economy and we provide a bound on the speed of convergence.[11] Our Theorem 4 then implies that Double Auctions in these markets also converge. As discussed in the Section 1, our convergence result does not rely on any continuity assumptions on distributions from which finite economies are drawn.

For simplicity and without loss of generality, we restrict our attention to equally scaled market sides, that is, we consider an infinite market $(\mathcal{B}, \mathcal{S}, v_B, v_S)$ with $\mu_B(\mathcal{B}) = 1$ and $\mu_S(\mathcal{S}) = 1$. We construct a random growing sequence of finite markets $(\mathcal{B}^n, \mathcal{S}^n v_B^n, v_S^n)$ from a limit market $(\mathcal{B}, \mathcal{S}, v_B, v_S)$ as follows: Consider a sequence of buy orders $b_1, b_2, ..., b_n$ and a sequence of sell orders $s_1, s_2, ..., s_n$. Note that $P \mapsto {}^{S_f(P)}\!/\!_\sigma$ and $P \mapsto 1 - {}^{D_f(P+)}\!/\!_\beta$ are cumulative distribution functions. The bids of buy orders and the asks of sell orders are drawn independently from distributions with these two cumulative distribution functions, which we denote by F_B for bids and F_S for asks.

Theorem 4 (Random economies and convergence rate). *A random growing sequence of finite markets $(\mathcal{B}^n, \mathcal{S}^n, v_B^n, v_S^n)$ generated by independent sampling from a limit market $(\mathcal{B}, \mathcal{S}, v_B, v_S)$ converges to the limit market a.s. at an exponential rate.*

We provide the proof in Appendix B.5.

5   Examples

In the following two finite markets we illustrate our definitions and results (including previewing some of the lemmas we introduce in the Appendix A). In both markets, there are two buy orders $\mathcal{B} = \{b_1, b_2\}$ and one sell order $\mathcal{S} = \{s_1\}$ such that $v_b^1 \geq v_b^2 > v_s^1$. The two markets differ in whether the buy orders include ties or not: in the market *with ties* $v_b^1 = v_b^2$, and in the market *without ties* $v_b^1 > v_b^2$.

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[11]Related insights for matching markets without transfers were established by Azevedo and Leshno (2016).

Demand and supply. Demand and supply functions are

$$D_f(P) = \begin{cases} 2 & \text{if } P \le v_b^2 \\ 1 & \text{if } v_b^2 < P \le v_b^1 \\ 0 & \text{if } P > v_b^1 \end{cases} \text{ and } S_f(P) = \begin{cases} 0 & \text{if } P < v_s^1 \\ 1 & \text{if } P \ge v_s^1. \end{cases}$$

For the supply correspondence, it follows from Lemma 2 that $S_c(P) = S_f(P)$ for $P \ne v_s^1$ and $S_c(v_s^1) = [0, 1]$. As mentioned above, the demand correspondence depends on whether ties exist or not. Without ties, it holds that $D_c(v_b^1) = [1, 2]$, $D_c(v_b^2) = [0, 1]$, and $D_c(P) = D_f(P)$ otherwise. With ties, it holds that $D_c(v_b^1) = D_c(v_b^2) = [0, 2]$, and again $D_c(P) = D_f(P)$ otherwise.

(Strong) market clearing prices. In the market without ties, the set of strong market clearing prices $\mathcal{P}_{SMC}$ is equal to the half-open interval $\mathcal{P}_{MC} = (v_b^2, v_b^1]$, while the set of market clearing prices $\mathcal{P}_{MC}$ is equal to the closed interval $[v_b^1, v_b^2]$. Therefore, in line with Lemma 6, $\mathcal{P}_{MC} = \overline{\mathcal{P}_{SMC}}$ holds. In the market with ties, there exists no strong market clearing price at all, that is, the set $\mathcal{P}_{SMC}$ is empty. However, there exists a unique market clearing price, that is $\mathcal{P}_{MC} = \{v_b^1 = v_b^2\}$. Therefore, again in line with Lemma 6, it holds that $\mathcal{P}_{MC}$ is a singleton.

Trade volume and excess. In the market without ties, any price in $(v_b^1, v_b^2]$ maximizes trade volume at 1 and minimizes excess at 0. But at v_b^1, the trade excess is equal to 1 and not minimized. In the market with ties, the unique market clearing price is $v_b^1 = v_b^2$, which maximizes trade volume at 1 and minimizes excess at 1. But every other price in $[v_s^1, v_b^1 = v_b^2)$ has the same trade volume and excess.

Walrasian equilibria. In the market without ties, any price $P \in [v_b^2, v_b^1]$ is a Walrasian equilibrium price with allocation $\mathcal{B}^* = \{b_1\}$ and $\mathcal{S}^* = \{s_1\}$. That is because this market outcome balances trade with one filled order on each market side. The allocation is individually rational, because $v_b^1 \ge P$ and $v_s^1 \le P$, and it is stable, because $v_b^2 \le P$. These are the only equilibrium outcomes, as any price above v_b^1 or below v_s^1 are not individually rational for buy order b_1 or sell order s_1, and any price in the interval $[v_s^1, v_b^2)$ leads to a market outcome that is not stable. The latter is true, because the unfilled buy order is strictly greater than P. Using similar reasoning in the market with ties, we find that the price $P = v_b^1 = v_b^2$ is the unique Walrasian equilibrium price with allocation $\mathcal{B}^* = \{b_1\}$ or $\mathcal{B}^* = \{b_2\}$

and $\mathcal{S}^* = \{s_1\}$. In line with Proposition 1, the set of equilibrium prices coincides with the set of market clearing prices that was computed above.

The Double Auction. In the market without ties, the set of market clearing prices is $[v_b^2, v_b^1]$. The Double Auction selects the market price according to some pricing rule. The pricing rule $P^* = kv_b^2 + (1-k)v_b^1$ corresponds to the k-Double Auction. Note that for $k = 0$, the Double Auction does not minimize excess. The allocation is $\mathcal{S}^* = \{s_1\}$ and $\mathcal{B}^* = \{b_1\}$. In the market with ties, the unique market clearing price is $P^* = v_b^2 = v_b^1$. The allocation is $\mathcal{S}^* = \{s_1\}$ and $\mathcal{B}^* = \{b_1\}$ or $\mathcal{B}^* = \{b_2\}$ (the latter chosen by a fair coin toss). The Double Auction, therefore, coincides with the k-Double Auction.

6 Conclusion

In this paper, we provide a unified definition of double auctions in finite and infinite markets. We prove that our Double Auction yields market clearing and Walrasian equilibrium, that the Double Auction nests as special cases the k-Double Auction for finite markets (cf. Rustichini et al. 1994) and continuous and strictly monotone demand and supply models (cf. Mas-Colell et al. 1995; Reny and Perry 2006), and we establish convergence of finite to infinite markets. Importantly, and in contrast to the previous infinite market literature, we do not impose any regularity assumptions. Such assumptions are satisfied in some but fail in other real-world applications, for example, when fees are present (Jantschgi et al., 2022).

Recently, in other settings canonical mechanisms such as deferred acceptance (Abdulkadiroğlu et al., 2015; Azevedo and Leshno, 2016), serial dictatorship (Che and Kojima, 2010), and top trading cycle (Leshno and Lo, 2020) were extended from finite to infinite markets, facilitating the analysis of these mechanisms. We hope that our unified Double Auction will have similar impact.

A Auxiliary lemmas

This section contains a series of lemmas that are used in the proofs of our main results. As discussed in Section 3, these lemmas are known in related models.

We include them for completeness. The proofs are in the Online Supplementary Appendix C.

Demand and supply. Our first lemmas concern demand and supply. First, we prove that demand and supply functions are analytically well-behaved.

Lemma 1 (Regularity of demand and supply functions). *The demand function $D_f(\cdot)$ is non-increasing, left-continuous and has right limits. The supply function $S_f(\cdot)$ is non-decreasing, right-continuous and has left limits. The limits can be expressed as $D_f(P+) = \mu_B(\mathcal{B}_>(P))$ and $S_f(P-) = \mu_S(\mathcal{S}_<(P))$.*

We then use Lemma 1 to express demand and supply correspondences.

Lemma 2 (Representation of demand and supply correspondences). *It holds that $D_c(P) = [D_f(P+), D_f(P)]$ and $S_c(P) = [S_f(P-), S_f(P)]$. Further:*

- *$D_f(P) = D_c(P) \Leftrightarrow \mu_B(\mathcal{B}_=(P)) = 0 \Leftrightarrow$ the demand function is continuous at P.*

- *$S_f(P) = S_c(P) \Leftrightarrow \mu_S(\mathcal{B}_=(\mathcal{S})) = 0 \Leftrightarrow$ the supply function is continuous at P.*

Market clearing. Next, we show that the set of market clearing prices $\mathcal{P}_{MC}$ can be expressed in terms of demand and supply functions.[12]

Lemma 3 ($\mathcal{P}_{MC}$ via D_f and S_f). *P is a market clearing price if and only if $D_f(P) \geq S_f(P)$ and $D_f(P+) \leq S_f(P)$ or $S_f(P) \geq D_f(P)$ and $S_f(P-) \leq D_f(P)$.*

The two cases describe on which side of the market there may be excess at price P. For some of our results, Lemma 3 can be used to determine useful bounds on the set $\mathcal{P}_{MC}$ in terms of demand and supply functions.

Definition (Lower and upper bounds). A price P is a *lower bound*, if for all $P' < P$ it holds that $D_f(P') > S_f(P')$ and an *upper bound*, if for all $P' > P$ it holds that $S_f(P') > D_f(P')$.

Lemma 4 (Bounds on $\mathcal{P}_{MC}$). *For any price P:*

- *If P is a lower bound then $P \leq \inf \mathcal{P}_{MC}$; if additionally $P \in \mathcal{P}_{MC}$, then $P = \min \mathcal{P}_{MC}$.*

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[12]Tatur (2005) mentions a similar construction, when demand and supply functions are given by probability distributions with jump-discontinuities.

- *If P is an upper bound, then $P \geq \sup \mathcal{P}_{MC}$; if additionally $P \in \mathcal{P}_{MC}$, then $P = \max \mathcal{P}_{MC}$.*

Next, we discuss the geometrical structure of the set of market clearing prices and, by Proposition 1, also of the set of Walrasian equilibrium prices.

Lemma 5 (Geometry of $\mathcal{P}_{MC}$). *The set of market clearing prices is non-empty, convex, and closed.*

A stronger definition of market clearing that is sometimes used colloquially requires demand and supply functions to be equal at a given price.

Definition (Strong market clearing price). A price P is a *strong market clearing price* if $D_f(P) = S_f(P)$. Denote the set of all strong market clearing prices by $\mathcal{P}_{SMC}$.

Our example in Section 5 shows that there might not exist a strong market clearing price. The following theorem discusses the geometric properties of the set of strong market clearing prices $\mathcal{P}_{SMC}$ and shows that the set of market clearing prices $\mathcal{P}_{MC}$ can be viewed as the minimal extension of $\mathcal{P}_{SMC}$ to guarantee existence.

Lemma 6 (Geometry of $\mathcal{P}_{SMC}$). *The set $\mathcal{P}_{SMC}$ is a convex subset of V. Every strong market clearing price is a market clearing price, that is $\mathcal{P}_{SMC} \subset \mathcal{P}_{MC}$. The set $\mathcal{P}_{MC} \setminus \mathcal{P}_{SMC}$ has Lebesgue-measure zero. Concretely, if $\mathcal{P}_{SMC} \neq \emptyset$, then $\mathcal{P}_{MC} = \overline{\mathcal{P}_{SMC}}$, and if $\mathcal{P}_{SMC} = \emptyset$, then $\mathcal{P}_{MC}$ is a singleton.*

In finite markets with m buy orders and n sell orders, Rustichini et al. (1994) show that any price in $(v^{(m)}, v^{(m+1)})$ is a strong market clearing price, hence:

Lemma 7 (Strong market clearing). *In finite markets, if $v^{(m)} \neq v^{(m+1)}$, then for $k \in (0,1)$ the k-Double Auction results in a strong market clearing price. That is, $\left(v^{(m)}, v^{(m+1)}\right) \subseteq \mathcal{P}_{SMC}$.*

However, if $k \in \{0,1\}$, the k-Double auction might not result in a strong market clearing price.[13] Moreover, if ties exist, the set of strong market clearing prices might be empty, see Section 5.[14]

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[13]These two mechanisms, called the *seller's* and *buyer's Double Auction*, are often studied separately from the case $k \in (0,1)$ (Satterthwaite and Williams, 1989b; Williams, 1991; Satterthwaite et al., 2014, 2022).

[14]Wilson (1985); Rustichini et al. (1994) acknowledge that if $v^{(m)} = v^{(m+1)}$, excess might exist and the k-Double Auction may require rationing.

We extend Lemma 7 to solidify the folk-wisdom that the k-Double Auction in finite markets results in and is fully characterized by market clearing, even in the case of ties.[15]

Lemma 8 (Market clearing)**.** *In finite markets, for $k \in [0,1]$, the k-Double Auction results in a market clearing price. Furthermore, $\left[v^{(m)}, v^{(m+1)}\right] = \mathcal{P}_{MC}$.*

Connection of market clearing to trade volume and excess. We show that market clearing implies that trade volume is maximized, and that strong market clearing implies that excess is minimized.

Lemma 9 ($\mathcal{P}_{MC}$ maximizes $Q(\cdot)$)**.** *A market clearing price maximizes the trade volume.*

Lemma 10 ($\mathcal{P}_{SMC}$ minimizes $Ex(\cdot)$)**.** *A strong market clearing price minimizes the excess.*

Maximizing trade volume and minimizing excess are the two appealing properties from a social welfare perspective, but they are not sufficient to characterize a double auction. On the one hand, there may exist prices that maximize trade volume and minimize excess but are not market clearing. On the other hand, there also may exist two market clearing prices with different excess. Section 5 provides examples of such markets.

Uniform Rationing in the Double Auction A standard rationing rule for Double Auctions is the uniform rationing rule. Recall that buy orders in $\tilde{\mathcal{B}}$ are selected uniformly at random from $\mathcal{B}_=(P^*)$ if the probability $\mathbb{P}[b \in \tilde{\mathcal{B}}]$ is the same for all $b \in \mathcal{B}_=(P^*)$. The random selection from $\mathcal{S}_=(P^*)$ is defined analogously.

Lemma 11 (Uniform Rationing)**.** *The uniform rationing rule is well-defined for any market instance in finite and infinite markets.*

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[15]This result is already present without a formal proof in the literature. Rustichini et al. (1994) justifies it with Lemma 7, and Cripps and Swinkels (2006) states that it can be seen after "a little time with the appropriate figure" of demand and supply schedules. Both do not provide a rigorous definition of market clearing prices.

B Proofs of the Main Results

B.1 Proof of Proposition 1

Proof. Recall that P is a market clearing price if and only if (i) $D_f(P) \geq S_f(P)$ and $D_f(P+) \leq S_f(P)$ or (ii) $S_f(P) \geq D_f(P)$ and $S_f(P-) \leq D_f(P)$ (Lemma 3). As a shorthand notation for our proofs, we will say that a market clearing price is of *type I*, if the first set of conditions holds, and of *type II*, if the second set of conditions hold.

First, we prove that $\mathcal{P}_{MC} \subset \mathcal{P}_{EQ}$. Consider that P^* is of type I, that is $D_f(P^*) \geq S_f(P^*)$ and $D_f(P^*+) \leq S_f(P^*)$. Set $\mathcal{S}^* = \mathcal{S}_{\leq}(P^*)$. Consider the set $\mathcal{B}_{>}(P^*)$. Lemma 1 shows that $D_f(P^*+) = \mu_B(\mathcal{B}_{>}(P^*))$. Let $x = S_f(P^*) - \mu_B(\mathcal{B}_{>}(P^*)) \geq 0$ and let $\tilde{\mathcal{B}}$ be a subset of $\mathcal{B}_{=}(P^*)$ with μ_B-measure equal to x. Such a set exists because $D_f(P^*) = \mu_B(\mathcal{B}_{\geq}(P^*)) = \mu_B(\mathcal{B}_{>}(P^*)) + \mu_B(\mathcal{B}_{=}(P^*)) \geq S_f(P^*)$ and $D_f(P^*+) = \mu_B(\mathcal{B}_{>}(P^*)) \leq S_f(P^*)$. Set $\mathcal{B}^* = \mathcal{B}_{>}P^* \cup \tilde{\mathcal{B}}$. We show that $(P^*, \mathcal{S}^*, \mathcal{B}^*)$ is a Walrasian equilibrium. It balances trade, because $\mu_B(\mathcal{B}^*) = \mu_B(\mathcal{B}_{>}(P^*)) + \mu_B((\tilde{\mathcal{B}})) = \mu_B(\mathcal{B}_{>}(P^*)) + S_f(P^*) - \mu_B(\mathcal{B}_{>}(P^*))$. Individual rationality follows, because $\mathcal{B}^* \subset \mathcal{B}_{\geq}(P^*)$ and $\mathcal{S}^* = \mathcal{S}_{\leq}(P^*)$. It is stable, because $\mathcal{B}_{>}(P^*) \subset \mathcal{B}^*$ and $\mathcal{S}_{<}(P^*) \subset \mathcal{S}^*$, as $\mathcal{B}^* = \mathcal{B}_{>}P^* \cup \tilde{\mathcal{B}}$ and $\mathcal{S}^* = \mathcal{S}_{\leq}(P^*)$. If there exists a market clearing price P^* of type II, that is $S_f(P^*) \geq D_f(P^*)$ and $S_f(P^*-) \leq D_f(P^*)$, one can construct the Walrasian equilibrium analogously.

Second, we show that $P \notin \mathcal{P}_{MC} \Rightarrow P \notin \mathcal{P}_{EQ}$. One of two cases must hold: (i) $D_f(P) > S_f(P)$ and $D_f(P-) > S_f(P)$ or (ii) $S_f(P) > D_f(P)$ and $S_f(P-) > D_f(P)$. For (i) there exists a price $P' > P$ with $D_f(P') > S_f(P)$. Assume that there exist sets $\mathcal{B}^*$ and $\mathcal{S}^*$, such that $(P, \mathcal{B}^*, \mathcal{S}^*)$ is a Walrasian equilibrium. Individual rationality implies that $\mathcal{B}^* \subset \mathcal{B}_{\geq}(P)$ and $\mathcal{S}^* \subset \mathcal{S}_{\leq}(P)$ hold. Next, stability implies that $\mathcal{B}_{>}(P) \subset \mathcal{B}^*$ and $\mathcal{S}_{<}(P) \subset \mathcal{S}^*$ hold. Those two inclusions imply that $\mu_B(\mathcal{B}_{\geq}(P)) \geq \mu_B(\mathcal{B}^*) \geq \mu_B(\mathcal{B}_{>}(P))$ and $\mu_S(\mathcal{S}_{\leq}(P)) \geq \mu_S(\mathcal{S}^*) \geq \mu_S(\mathcal{S}_{<}(P))$.

For a price $P' > P$ it holds that $\mu_B(\mathcal{B}_{>}(P)) \geq \mu_B(\mathcal{B}_{\geq}(P'))$. Hence, $\mu_B(\mathcal{B}^*) \geq \mu_B(\mathcal{B}_{>}(P)) \geq \mu_B(\mathcal{B}_{\geq}(P')) = D_f(P') > S_f(P) \geq \mu_S(\mathcal{S}^*)$. This proves that $(P, \mathcal{B}^*, \mathcal{S}^*)$ is not a Walrasian equilibrium, because it does not balance trade. For (ii), the proof is analogous. $\square$

B.2 Proof of Theorem 1

Proof. We show in Lemma 5 in Appendix A that the set of market clearing prices $\mathcal{P}_{MC}$ is non-empty, compact, and convex, which implies that the interval $[\min \mathcal{P}_{MC}, \max \mathcal{P}_{MC}]$ is well-defined. By Proposition 1 this set coincides with $\mathcal{P}_{EQ}$, proving that the Double Auction results in market clearing and Walrasian equilibrium. Lemma 9 in Appendix A proves that every market clearing price maximizes the trade volume. It is shown in the proof of Proposition 1 that an allocation exists with $\mathcal{S}^* = \mathcal{S}_{\leq}(P^*) \cup \tilde{S}$ and $\mathcal{B}^* = \mathcal{B}_{>}(P^*) \cup \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}} \subset \mathcal{B}_{=}(P^*)$, $\tilde{S} \subset \mathcal{S}_{=}(P^*)$ and at least one of the two sets is empty. $\qquad\square$

B.3 Proof of Theorem 2

Proof. For finite markets, it is proven in Lemma 8 in Appendix A that the set of market clearing prices $\mathcal{P}_{MC}$ is equal to the interval $\left[v^{(m)}, v^{(m+1)}\right]$. Therefore, the price-setting rule $P^* = k \cdot \min \mathcal{P}_{MC} + (1 - k) \cdot \max \mathcal{P}_{MC}$ leads to the market price chosen by the k-Double Auction. Moreover, the allocation rules of the Double Auction and the k-Double Auction coincide by construction. In infinite markets with continuous and strictly monotone demand and supply functions, by assumption, there exists a market clearing price P^* with $D_f(P^*) = S_f(P^*)$. Because demand and supply functions are continuous, Lemma 2 in Appendix A implies that $D_f(P) = D_c(P)$ and $S_f(P) = S_c(P)$ for all $P \in V$. The strict monotonicity of demand and supply functions then implies that this market clearing price is unique, that is $\mathcal{P}_{MC} = \{P^*\}$. This proves that the Double Auction always chooses P^* as the market price. Moreover, it holds that $Ex(P^*) = 0$. Therefore, by construction, the Double Auction results in the allocation $\mathcal{B}^* = \mathcal{B}_{\geq}(P^*)$ and $\mathcal{S}^* = \mathcal{S}_{\leq}(P^*)$, which coincides with the allocation in the textbook models of continuous and strictly monotone demand and supply. $\quad\square$

B.4 Proof of Theorem 3

Proof. To prove that $\lim_{k \to \infty} d(\min \mathcal{P}_{MC}^k, \mathcal{P}_{MC}) = 0$, we show that for any $P < \min \mathcal{P}_{MC}$ there exists a market size k_0, such that for all $k \geq k_0$ it holds that $\min \mathcal{P}_{MC}^k > P$. To show that $\min \mathcal{P}_{MC}^k > P$, Lemma 4 in Appendix A implies that it suffices to show that there exists $\gamma > 0$ with $D_f^k(P + \gamma) > S_f^k(P + \gamma)$. Because $P < \min \mathcal{P}_{MC}$ by assumption, there exists $\gamma > 0$ with $D_f(P + \gamma) >$

$S_f(P + \gamma)$. Let $\alpha > 0$ be the difference of demand and supply at $P + \gamma$, that is $\alpha = D_f(P + \gamma) - S_f(P + \gamma)$. It follows from the uniform convergence of demand and supply, that for any $\epsilon > 0$ there exists k_0 such that for all $k \geq k_0$ $|D_f(P + \gamma) - D_f^k(P + \gamma)| \leq \epsilon$ and $|S_f(P + \gamma) - S_f^k(P + \gamma)| \leq \epsilon$. For such k, it holds that $D_f^k(P + \gamma) - S_f^k(P + \gamma) \geq D_f(P + \gamma) - S_f(P + \gamma) - 2\epsilon = \alpha - 2\epsilon$. If $\epsilon > 0$ was chosen such that $\epsilon < {}^\alpha/_2$, then $D_f^k(P + \gamma) - S_f^k(P + \gamma) > 0$, which shows that $\min \mathcal{P}_{MC}^k > P$. Note that this also implies that $\max \mathcal{P}_{MC}^k > P$. It can be proven analogously that for any $P > \max \mathcal{P}_{MC}$ there exists a market size k_0, such that for all $k \geq n_0$ it holds that $\max \mathcal{P}_{MC}^k < P$, which also implies that $\min \mathcal{P}_{MC}^k < P$. Combining these bounds yields that $\lim_{k \to \infty} d(\min \mathcal{P}_{MC}^k, \mathcal{P}_{MC}) = 0$ and $\lim_{k \to \infty} d(\max \mathcal{P}_{MC}^k, \mathcal{P}_{MC}) = 0$. $\qquad \square$

B.5 Proof of Theorem 4

Proof. Let $n \geq 1$ denote the number of buy orders and the number of sell orders. The bids of buy orders are n independent random variables $v_b^1, ..., v_b^n$ with cumulative distribution function $F_B(P) = 1 - {}^{D_f(P+)}/_\beta$. The asks of sell orders are n independent random variables $v_s^1, ..., v_s^n$ with cumulative distribution function $F_S(P) = {}^{S_f(P)}/_\sigma$. For $A \subset V$, define $\mu_B^{v,n}(A) = \sum_{i=1}^n \delta_{v_b^i}(A)/n$ and $\mu_S^{v,n}(A) = \sum_{j=1}^n \delta_{v_s^j}(A)/n$. Demand and supply functions are then the random variables $D_f^n(P) = \mu_B^{v,n}([P, \overline{v}])$ and $S_f^n(P) = \mu_S^{v,n}([\underline{v}, P])$. The Glivenko-Cantelli Theorem implies that $|D_f^n - D_f|_\infty \to 0$ and $|S_f^n - S_f|_\infty \to 0$ almost surely. Hence, the random growing sequence of finite markets with random value measures $\mu_B^{v,n}$ and $\mu_S^{v,n}$ obtained by independent sampling converges almost surely to $(\mathcal{B}, \mathcal{S}, \mu_B^v, \mu_S^v)$. It follows from the Vapnis-Chervonenkis Theorem,[16] that for fixed $\epsilon > 0$, there exist constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that $\mathbb{P}\left[\sup_{P \in \mathbb{R}} |D_f^n(P) - D_f(P)| > \epsilon\right] \leq \alpha_1 e^{-\beta_1 n}$, $\mathbb{P}\left[\sup_{P \in \mathbb{R}} |S_f^n(P) - S_f(P)| > \epsilon\right] \leq \alpha_2 e^{-\beta_2 n}$. As the bids and asks are independent, the random variables D_f^n and S_f^n are independent as well. Hence,

$$\mathbb{P}\left[\sup_{P \in \mathbb{R}} |S_f^n(P) - S_f(P)| > \epsilon \land \sup_{P \in \mathbb{R}} |S_f^n(P) - S_f(P)| > \epsilon\right] \leq \alpha e^{-\beta n}$$

holds with $\alpha = \alpha_1 \cdot \alpha_2$ and $\beta = \beta_1 + \beta_2$. $\qquad \square$

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[16]See Vapnik and Chervonenkis (1971) and Devroye et al. (2013). The important requirement for the theorem to apply is that the Vapnik-Chervonenkis dimensions of the class of sets $\{v \in V : v \leq P\}$ and $\{v \in V : v \geq P\}$ are finite.

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C   Online Supplementary Appendix

Proofs of the Auxiliary Lemmas in Appendix A

Proof of Lemma 1. Let μ_S^v be the pushforward measures of μ_S with respect to v_S, that is $\mu_S^v(\cdot) = \mu_S(v_S^{-1}(\cdot))$. μ_S^v is σ-additive and finite on V. Denote by $\underline{V}(P)$ the interval $[\underline{v}, P]$. It holds that $S_f(P) = \mu_S^v(\underline{V}(P))$. Moreover, if $P_1 > P_2$, then $\underline{V}(P_2) \subset \underline{V}(P_1)$. The σ-additivity of μ_S^v yields

$$S_f(P_1) = \mu_S^v(\underline{V}(P_1)) \geq \mu_S^v(\underline{V}(P_2)) = S_f(P_1),$$

which proves that $S_f(\cdot)$ is non-decreasing.

Every monotonic function has limits from the right and the left for every point in its domain. Next, consider a strictly decreasing sequence of prices $P_n \downarrow P$. $\underline{V}(P_n)$ is a decreasing sequence of sets, that is $\underline{V}(P_{n+1}) \subset \underline{V}(P_n)$. It holds that $\lim_{n \to \infty} \underline{V}(P_n) = \bigcap_{n=1}^{\infty} \underline{V}(P_n) = \underline{V}(P)$. As a finite measure on $\mathbb{R}$, μ_S^v is continuous from above, see e.g., Folland (1999). That is if $\{A_i\}_i \subset V$ is a sequence of sets with $A_1 \supset A_2 \supset A_3 \supset ...$, then $\mu_S^v\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu_S^v(A_i)$. This yields

$$\lim_{P_n \downarrow P} S_f(P_n) = \lim_{n \to \infty} \mu_S^v(\underline{V}(P_n)) = \mu_S^v\left(\bigcap_{n=1}^{\infty} \underline{V}(P_n)\right) = \mu_S^v(\underline{V}(P)) = S_f(P),$$

which proves the right-continuity of $S_f(\cdot)$.

To show that $S_f(P-) = \mu_S(\mathcal{S}_<(P)) = \mu_S^v([\underline{v}, P))$, note μ_S^v is a σ-additive Borel-measure on $\mathbb{R}$ and therefore regular, see Bogachev (2007). That is, for all Borel-sets A $\mu_S^v(A) = \sup\{\mu_S^v(F)|F \subset A, F \text{ compact, Borel}\}$. It is therefore sufficient to approximate the interval $[\underline{v}, P)$ by compact sets $[\underline{v}, P']$ with $P' < P$. It finally holds that

$$\sup\{\mu_S^v(F)|F \subset A, F \text{ compact, Borel}\} = \lim_{P' \uparrow P} \mu_S^v(([\underline{t}, P']) = \lim_{P' \uparrow P} S_f(P') = S_f(P-),$$

which implies that $S_f(P-) = \mu_S(\mathcal{S}_<(P))$. The proof that demand is non-increasing, left-continuous, and has right limits, as well as $D_f(P+) = \mu_B(\mathcal{B}_>(P))$ is analogous. $\square$

Proof of Lemma 2. Recall that $D_f(P) = \mu_B(\mathcal{B}_\geq(P))$ and $S_f(P) = \mu_S(\mathcal{S}_\leq(P))$. It follows from Lemma 1 that $D_f(P+) = \mu_B(\mathcal{B}_>(P))$ and $S_f(P-) = \mu_S(\mathcal{S}_<(P))$,

which implies

$$D_c(P) = [\mu_B(\mathcal{B}_>(P)), \mu_B(\mathcal{B}_\geq(P))] = [D_f(P+), D_f(P)] \text{ and}$$
$$S_c(P) = [\mu_S(\mathcal{S}_<(P)), \mu_S(\mathcal{S}_\leq(P))] = [S_f(P-), S_f(P)].$$

The σ-additivity of μ_B implies that

$$D_f(P) = \mu_B(\mathcal{B}_\geq(P)) = \mu_B(\mathcal{B}_>(P)) + \mu_B(\mathcal{B}_=(P)) = D_f(P+) + \mu_B(\mathcal{B}_=(P)).$$

Because $D_f(P) = D_c(P)$ if and only if $D_f(P+) = D_f(P)$, this is equivalent to $\mu_B(\mathcal{B}_=(P)) = 0$. Furthermore, $D_f(P+) = D_f(P)$, that is, D_f is right-continuous at P. As $D_f(\cdot)$ is left-continuous by Lemma 1, $D_f(P) = D_c(P)$ is equivalent to continuity at P. The proof for supply is analogous. $\qquad\square$

Proof of Lemma 3. If $D_f(P) \geq S_f(P)$, then $D_c(P) \cap S_c(P) \neq \emptyset \Leftrightarrow S_f(P-) \geq D_f(P)$. If $D_f(P) \leq S_f(P)$, then $D_c(P) \cap S_c(P) \neq \emptyset \Leftrightarrow D_f(P+) \geq S_f(P)$. $\qquad\square$

Proof of Lemma 4. Consider that P is a lower bound. It suffices to prove that for a price $P' < P$ $P' \notin \mathcal{P}_{MC}$ holds. If $P \in \mathcal{P}_{MC}$, it follows directly that $P = \min \mathcal{P}_{MC}$. Because $D_f(P') > S_f(P')$, it is sufficient to prove that $D_f(P'-) > S_f(P')$. For P'' in (P', P) it holds that $D_f(P'') > S_f(P'')$. The monotonicity of $D_f(\cdot)$ and $S_f(\cdot)$ yields

$$D_f(P'-) \geq D_f(P'') > S_f(P'') \geq S_f(P').$$

If P is an upper bound, the proof is analogous. $\qquad\square$

Proof of Lemma 5. First, we show that $\mathcal{P}_{MC}$ is non-empty. Consider the set $\hat{\mathcal{P}} = \{P : D_f(P) \geq S_f(P)\}$. The monotonicity of $D_f(\cdot)$ and $S_f(\cdot)$ yields that $\hat{\mathcal{P}}$ is convex. For $P < \underline{v}$ it holds that $D_f(P) = \mu_B(\mathcal{B}) > 0 = S_f(P)$, proving that $\hat{\mathcal{P}}$ is non-empty. For $P > \overline{v}$ it holds that $D_f(P) = 0 < \mu_S(\mathcal{S}^*) = S_f(P)$, proving that $\hat{\mathcal{P}}$ is bounded from above. For such a set the supremum $P^* = \sup \hat{\mathcal{P}}$ exists and is unique. We show that $P^* \in \mathcal{P}_{MC}$. Two cases need to be considered separately: (i) $P^* \in \hat{\mathcal{P}}$ and (ii) $P^* \notin \hat{\mathcal{P}}$.

For (i), $D_f(P^*) \geq S_f(P^*)$ and for all $P' > P^*$ $D_f(P') < S_f(P')$. The right-continuity of $S_f(\cdot)$ implies $D_f(P'+) \leq S_f(P')$, which proves that P is a market clearing price of type I.

For (ii), $D_f(P^*) < S_f(P^*)$ holds. The monotonicity of $D_f(\cdot)$ and $S_f(\cdot)$ implies that for all $P' < P^*$ $D_f(P') \geq S_f(P')$ holds. Left-continuity of $D_f(\cdot)$ yields that $D_f(P') \geq S_f(P'-)$, which proves that P^* is a market clearing price of type II. We will show below in the proof of Lemma 6 that the set of strong market clearing prices $\mathcal{P}_{SMC}$ is convex. The same proof proceeds to show that if $\mathcal{P}_{SMC} \neq \emptyset$, then $\mathcal{P}_{MC} = \overline{\mathcal{P}_{SMC}}$, and if $\mathcal{P}_{SMC} = \emptyset$, then $\mathcal{P}_{MC}$ is a singleton. This proves that $\mathcal{P}_{MC}$ is convex and closed. $\qquad\square$

Proof of Lemma 6. We first show that $\mathcal{P}_{SMC}$ is convex. As the empty set is convex by convention, assume that $\mathcal{P}_{SMC} \neq \emptyset$. Consider $P_1, P_2 \in \mathcal{P}_{SMC}$ with $P_1 \leq P_2$. The monotonicity of $D_f(\cdot)$ and $S_f(\cdot)$ implies that $D_f(P_1) \geq D_f(P_2) = S_f(P_2) \geq S_f(P_1)$, which proves that $D_f(\cdot)$ and $S_f(\cdot)$ are constant on $[P_1, P_2]$. Therefore, for any price $P \in [P_1, P_2]$ $P \in \mathcal{P}_{SMC}$ holds. For $P < \underline{v}$ $D_f(P) = \mu_B(\mathcal{B}) > 0 = S_f(P)$ and for $P > \overline{v}$ $D_f(P) = 0 < \mu_S(\mathcal{S}^*) = S_f(P)$ holds, which implies that $\mathcal{P}_{SMC} \subset V$.

Next, we show that $\mathcal{P}_{SMC} \subset \mathcal{P}_{MC}$. $P \in \mathcal{P}_{SMC} \Leftrightarrow D_f(P) = S_f(P)$. Because $D_f(\cdot)$ is non-increasing it follows that $D_f(P+) \geq D_f(P) = S_f(P)$, which proves that P is a market clearing price of type I. Because $S_f(\cdot)$ is non-decreasing it follows that $S_f(P-) \leq S_f(P) = D_f(P)$, which proves that P is a market clearing price of type II.

Next, assume that $\mathcal{P}_{SMC} = \emptyset$. To prove that P^* from the proof of Lemma 5 above is the unique market clearing price, it suffices to prove by Lemma 4 that P^* is both a lower and upper bound. P^* is either of type I, that is $D_f(P^*) > S_f(P^*)$ and $D_f(P^*+) \leq S_f(P^*)$ or of type II, that is $S_f(P^*) > D_f(P^*)$ and $S_f(P^*-) \leq D_f(P^*)$. It follows from monotonicity of $D_f(\cdot)$ and $S_f(\cdot)$ and the emptyness of $\mathcal{P}_{MC}$ that for all $P' < P$ it holds that $D_f(P') > S_f(P')$ and for all $P' > P$ it holds that $D_f(P') < S_f(P')$. Therefore P^* is indeed both a lower and upper bound.

Finally, assume that the interval $\mathcal{P}_{SMC} \neq \emptyset$. To show that $\mathcal{P}_{MC} = \overline{\mathcal{P}_{SMC}}$, by Lemma 4 it suffices to prove that $\underline{P} = \inf \mathcal{P}_{SMC}$ is both a market clearing price and lower bound and $\overline{P} = \sup \mathcal{P}_{SMC}$ is both a market clearing price and an upper bound. $D_f(\underline{P}) \geq S_f(\underline{P})$ by monotonicity of $D_f(\cdot)$ and $S_f(\cdot)$. By definition, for every P with $\overline{P} > P > \underline{P}$ it holds that $D_f(P) = S_f(P)$. It follows from the left continuity of $S_f(\cdot)$ that $D_f(\underline{P}+) = S_f(\underline{P})$, which proves that $\underline{P} \in \mathcal{P}_{MC}$. For every $P' < \underline{P}$ we have that $D_f(P') > S_f(P')$. Therefore $\underline{P}$ is a lower bound. Similar arguments yield that $\overline{P} \in \mathcal{P}_{MC}$ is an upper bound. $\qquad\square$

Proof of Lemma 7. We show that $P \in (v^{(m)}, v^{(m+1)}) \Rightarrow P \in \mathcal{P}_{SMC}$. Suppose

that $D_f(P) = k \geq 0$. It holds that $D_f(P) = D_f(v^{(m+1)})$ and $S_f(P) = S_f(v^{(m)})$. The set $\{v^{(m+1)}, ..., v^{(m+n)}\}$ has cardinality n and the number of bids in it is $D_f(v^{(m+1)}) = k$. Hence, the number of asks in it is $n - k$. As there is a total number of n asks, the number of asks in the set $\{v^{(1)}, ..., v^{(m)}\}$ is k. As this number is equal to $S_f(v^{(m)})$, it holds that $S_f(P) = k = D_f(P)$, which shows that $P \in \mathcal{P}_{SMC}$. $\qquad\square$

Proof of Lemma 8. There are two cases: (i) $v^{(m)} \neq v^{(m+1)}$, and (ii) $v^{(m)} = v^{(m+1)}$.

For (i), Lemma 7 shows that $P \in (v^{(m)}, v^{(m+1)}) \Rightarrow P \in \mathcal{P}_{SMC}$, and hence by Lemma 6 $P \in \mathcal{P}_{MC}$. Next, consider $v^{(m)}$. $S_f(v^{(m)}) = S_f(P)$ for P in $(v^{(m)}, v^{(m+1)})$. If there is a bid equal to $v^{(m)}$, it holds that $D_f(v^{(m)}) > D_f(P)$. If not, then $D_f(v^{(m)}) = D_f(P)$. This shows that $D_f(v^{(m)}) \geq D_f(P)$. Hence,

$$D_f(v^{(m)}) \geq D_f(P) = S_f(P) = S_f(v^{(m)}).$$

To show that $v^{(m)} \in \mathcal{P}_{MC}$, it is by Lemma 2 sufficient to show that $D_f(v^{(m)}+) \leq S_f(v^{(m)})$. $D_f(v^{(m)}+) = D_f(P)$ holds, as there are no bids or asks in $(v^{(m)}, v^{(m+1)})$. Therefore, $D_f(v^{(m)}+) = D_f(P) = S_f(P) = S_f(v^{(m)})$. A similar argument shows that $v^{(m+1)} \in \mathcal{P}_{MC}$. Finally, we show that $v^{(m+1)}$ is an upper bound and $v^{(m)}$ is a lower bound, which, by Lemma 4, implies that $v^{(m)} = \min \mathcal{P}_{MC}$ and $v^{(m+1)} = \max \mathcal{P}_{MC}$, which finishes the proof for (i). Consider $P > v^{(m+1)}$. $D_f(P) < S_f(P)$ holds, as demand decreases or supply increases due to bids and asks at $v^{(m+1)}$. Therefore $v^{(m+1)}$ is an upper bound. Similar arguments yield that $v^{(m)}$ is a lower bound.

For (ii), write $v = v^{(m)} = v^{(m+1)}$ for ease of notation. We will show $v \in \mathcal{P}_{MC}$ and that this price is both a lower and upper bound. Lemma 4 then implies that $\mathcal{P}_{MC} = \{v\}$. For a relation $\mathcal{R} \in \{\geq, >, =, <, \leq\}$, denote by $v_{\mathcal{R}}$ the number of bids and asks (strictly) above, below or equal to v. Denote by $v_{\mathcal{R},B}$ and $v_{\mathcal{R},S}$ the restriction to either bids or asks. It holds that

$$v_< \leq m - 1 \, , \, v_= \geq 2 \, , \, v_> \leq n - 1, \text{ ,and } v_< + v_= + v_> = m + n.$$

Note that $D_f(v) = v_{\geq,B} = v_{=,B} + v_{>,B} \geq 1$. That is because there are at most $m - 1$ bids and asks strictly below v and there is a total of m bids, which proves that at least one bid is greater or equal to v. Because the total number of asks is

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n,

$$S_f(v) = v_{\leq,S} = n - v_{>,S} = n - v_> + v_{>,B}.$$

Next, we prove that v is an upper bound. Consider $P > v$. $D_f(P) \leq v_{>,B}$ holds, because $v_{=,B}$ bids at v are lost and $S_f(P) \geq n - v_> + v_{>,B}$, because supply is non-decreasing. Hence, $S_f(P) - D_f(P) \geq n - v_>$. $v_> \leq n - 1$ implies that $S_f(P) - D_f(P) \geq 1$, which yields $S_f(P) > D_f(P)$.

Finally, we prove that v is a lower bound. Consider $P < v$. It holds that $S_f(P) \leq n - v_> + v_{>,B} - v_= + v_{=,B}$, because $v_= - v_{=,B}$ asks at v are lost and $D_f(P) \geq v_{=,B} + v_{>,B}$, because demand is non-increasing. This implies that $D_f(P) - S_f(P) \geq v_= + v_> - n$. But it follows from $v_< + v_= + v_> = m + n$ and $v_< \leq m - 1$ that $v_= + v_> = m + n - v_< \geq n + 1$, which implies that $D_f(P) - S_f(P) \geq 1 > 0$. Therefore, $D_f(P) > S_f(P)$. $\qquad\square$

Proof of Lemma 9. Consider $P \in \mathcal{P}_{MC}$. Assume that P is of type I, that is, $D_f(P) \geq S_f(P)$ and $D_f(P+) \leq S_f(P)$. Then $Q(P) = \min(D_f(P), S_f(P)) = S_f(P)$ holds. For any price $P' < P$, $S_f(P') \leq S_f(P)$ holds, because $S_f(\cdot)$ is non-decreasing. Therefore $Q(P') \leq Q(P)$. For any price $P'' > P$, it holds that $D(P'') \leq D(P+)$, because $D_f(\cdot)$ is non-increasing. $S_f(P) \geq D_f(P+)$ implies that $S_f(P) \geq D_f(P'')$. Therefore $Q(P'') \leq Q(P)$, which proves that P maximizes the trade volume. The proof for a market clearing price of type II proceeds in analogy. $\qquad\square$

Proof of Lemma 10. $P \in \mathcal{P}_{SMC} \Leftrightarrow D_f(P) = S_f(P)$ and hence $Ex(P) = 0$. $\qquad\square$

Proof of Lemma 11. In finite markets, it suffices to show that for a discrete set V with x elements, it is possible to select a subset $\tilde{V}$ with $y < x$ elements, such that the probability of being selected is the same for each element. If $y = 0$, set $\tilde{V} = \emptyset$. Otherwise, order $V = \{v_1, ..., v_x\}$. Set $\tilde{V} = \{\tilde{v}_1, ...\tilde{v}_y\}$, where $\tilde{v}_i$ is chosen uniformly at random from the set $\tilde{V}_i = V \setminus \{\tilde{v}_1, ..., \tilde{v}_{i-1}\}$ with probability $\frac{1}{x-(i-1)}$, where $\tilde{V}_1 = V$. It follows that for any $v \in V$, it holds that $\mathbb{P}[v \in \tilde{V}] = \frac{y}{x}$.

In infinite markets, we consider a bounded Borel set $V \subset \mathbb{R}$, such that its Lebesgue measure $\lambda(V)$ is equal to x. We prove that for any $y < x$ it is possible to select a subset $\tilde{V}$ with Lebesgue measure y, such that each point of V is selected with the same probability. Denote by $\bar{v}$ and $\underline{v}$ the supremum and infimum of V.

For any Borel set, there exists a uniform probability distribution with density equal to $1/x$ on V. Using this distribution select a point t_0 uniformly at random. For $0 \leq \epsilon \leq \overline{v} - \underline{v}$, define the following set

$$
V_{v_0}^\epsilon = \begin{cases} [v_0, v_0 + \epsilon] \cap V & \text{for } \epsilon \leq \overline{v} - v_0 \\ ([v_0, \overline{v}] \cap V) \cup ([\underline{v}, \underline{v} + \epsilon - (\overline{v} - v_0)] \cap V) & \text{for } \overline{v} - v_0 < \epsilon \leq \overline{v} - \underline{v} \end{cases}.
$$

Intuitively, the set $V_{v_0}^\epsilon$ is constructed by starting at the randomly selected point v_0 and including all points of V to its right, until the distance ϵ is reached. If the supremum of V is reached, the construction proceeds at the infimum of V. Define the function $f_{v_0}(\epsilon) = \lambda(V_{v_0}^\epsilon)$. This function is well-defined for $\epsilon \in [0, \overline{v} - \underline{v}]$, it is nondecreasing and continuous, and it holds that $f_{v_0}(0) = 0$ and $f_{v_0}(\overline{v} - \underline{v}) = x$. It follows from the Intermediate Value Theorem, that there exists $\tilde{\epsilon}$ with $f_{v_0}(\tilde{\epsilon}) = y$. Because $f_{v_0}(\cdot)$ is non-decreasing, this set is convex. Let ϵ^* be the infimum of this set. Because $f_{v_0}(\cdot)$ is continuous, it holds that $f_{v_0}(\epsilon^*) = f_{v_0}(\lim_{\epsilon \downarrow \epsilon^*} \epsilon) = \lim_{\epsilon \downarrow \epsilon^*} f_{v_0}(\epsilon) = y$. Therefore the set $V_{v_0}^{\epsilon^*}$ is a random set with Lebesgue measure y. Finally, it is a straight-forward computation that for all $v \in V$ it holds that $\mathbb{P}[v \in V_{v_0}^{\epsilon^*}] = \frac{x}{y} \equiv const.$ $\qquad \square$