Double Auctions and Walrasian Equilibrium

Simon Jantschgi† Heinrich H. Nax‡ Bary S.R. Pradelski§ Marek Pycia¶

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Abstract

We define a double auction mechanism, treating in a unified way finite and infinite markets, allowing for ties in reported values, and not imposing any regularity assumptions. It is the first such definition. In all markets, our Double Auction implements market clearing and a Walrasian equilibrium. In finite markets our Double Auction nests as special cases the standard $k$-Double Auction and in infinite markets the textbook model of continuous and strictly monotone demand and supply. Finally, we establish the convergence of finite to infinite Double Auctions.

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†Univ. Zurich, 8050 Zurich, Switzerland; simon.jantschgi@uzh.ch
‡Univ. Zurich, 8050 Zurich, Switzerland; heinrich.nax@uzh.ch
§Univ. Grenoble Alpes, CNRS, Inria, Grenoble INP, LIG, 38000 Grenoble, France; bary.pradelski@cnrs.fr
¶Univ. Zurich, 8006 Zurich, Switzerland; marek.pycia@econ.uzh.ch

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1 Introduction

Double auctions are simple canonical mechanisms that are ubiquitous in many asset and commodity markets.¹ In a double auction, potential buyers and sellers submit buy and sell orders to a central clearinghouse that then establishes who deals with whom and at what price. This price is commonly characterized as being market clearing and supporting an allocation equilibrating (revealed) demand and supply, thus implementing Walrasian equilibrium.² For markets with finitely many traders, the standard double auction mechanism is the $k$-Double Auction, as defined by Wilson (1985) and Rustichini et al. (1994),³ who provide explicit formulae for calculating market clearing prices.⁴ In the absence of ties they show that the market clearing prices balance demand and supply, implicitly proving the equivalence to Walrasian equilibrium. Approximating large finite markets, models with infinitely many traders are often used as they permit simpler analysis (cf. Aumann 1964).

The existent analyses of infinitely large double auctions impose regularity assumptions (such as continuity and strict monotonicity of demand and supply) to ensure tractability (cf. Mas-Colell et al. (1995) for a textbook model of continuous and strictly monotone demand and supply, and Reny and Perry (2006) who focus on an environment and equilibria in which these assumptions fail only with probability 0).⁵ These regularity assumptions, which guarantee that there exists a price that balances supply and demand, do not hold however in finite markets and they might also fail with positive probability in many continuum markets: the failure of these assumptions is related to excess demand or supply that arise when several traders submit the same bid as recognized already by Rustichini et al. (1994). The failure might be driven by the discreteness of the currency or by the presence of large buyers or sellers, which induces partial executions. The failure can also be driven by strategic incentives (Jackson et al., 2002; Woodward, 2019; Jantschgi et al., 2022) or by boundedly rational behavior.

¹See, for example, Friedman and Rust (1993) for an overview of double auctions in theory and practice.
²We shall use the Walrasian equilibrium terminology, also referred to as Walrasian competitive equilibrium, competitive equilibrium, or simply equilibrium.
³The two formulations are equivalent; we shall use the formulation by Rustichini et al. (1994).
⁴See also earlier formulations for the bilateral $k$-double auction case by Chatterjee and Samuelson (1983) and ?.
⁵See also Tatur (2005) who introduces regularity assumptions in large finite markets to facilitate the analysis.
and imitation (Banerjee, 1992; Shiller, 2000). At the same time, the explicit formulae that calculate market clearing prices in finite $k$-Double Auctions do not extend to infinite markets. See the table below for a summary of the prior state of knowledge.

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<th>Finite markets</th>
<th>Convergence</th>
<th>Infinite markets</th>
</tr>
</thead>
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<tr>
<td><strong>Mechanism</strong></td>
<td>$W85, RSW94$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
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<td><strong>Walrasian link</strong></td>
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$^a$For finite markets, this means an absence of ties. For infinite markets, Mas-Colell et al. (1995) treats the textbook model of continuous and strictly monotone demand and supply. Reny and Perry (2006) focus on supply and demand arising from equilibrium play with additional assumptions about their analytical properties and discrete bids and asks.

$^b$These papers show that there exists a market clearing price equating demand and supply, which essentially provides a link to Walrasian equilibrium.

In this article, we make the following contributions. We provide a unified definition of double auctions that works in both finite and infinite markets and does not rely on any regularity assumptions. Our mechanism implements market clearing and Walrasian equilibrium (cf. Theorem 1). Our Double Auction nests earlier formulations including the $k$-Double Auction in finite markets and continuous and strictly monotone demand and supply models in infinite markets (cf. Theorem 2). Finally, we establish convergence of finite to infinite markets (cf. Theorems 3 and 4). Our unified theory of Double Auctions addresses all open questions from the table above.

Our paper belongs to the recent wave of papers extending canonical mechanisms from finite to infinite markets: Abdulkadiroğlu et al. (2015) and Azevedo and Leshno (2016) extended deferred acceptance, Che and Kojima (2010) extended...
probabilistic serial and random serial dictatorship, and Leshno and Lo (2020) extended top trading cycle mechanisms. In terms of results structure, that is, providing a unified framework for finite and infinite markets and a convergence result that formally connects the two, our results on the Double Auction are particularly similar to those by Azevedo and Leshno (2016) for matching markets.

As discussed above, our paper contributes to the literature on double auctions and their definitions. The main thrust of the theoretical literature focused on large double auctions, asymptotic truthfulness and resulting efficiency, see, e.g., Rustichini et al. (1994), Jackson and Swinkels (2005), Cripps and Swinkels (2006), Reny and Perry (2006), and Jantschgi et al. (2022).

2 The market

We study two-sided markets with traders interested in either buying or selling a finite number of indivisible units of a good. Each trader can submit multiple single-unit buy or sell orders. We study double auctions, market clearing, and Walrasian equilibria taking these orders as primitives.\footnote{One can alternatively identify each order with a trader; this identification would be without loss of generality as we do not study incentive issues in this paper.}

Denote by $\mathcal{B} \subset \mathbb{R}$ the set of buy orders $(b \in \mathcal{B})$ and by $\mathcal{S} \subset \mathbb{R}$ the set of sell orders $(s \in \mathcal{S})$. Consider two cases: a market with finite numbers of orders on each market side (finite market) and a market with continuum of orders on each market side (infinite market). When the market has $m$ buy orders and $n$ sell orders, label orders as $\mathcal{B} = \{1, 2, \ldots, m\}$ and $\mathcal{S} = \{1, 2, \ldots, n\}$. When the market has a continuum of orders, the labels are real numbers from closed intervals $\mathcal{B} \subset \mathbb{R}$ and $\mathcal{S} \subset \mathbb{R}$. Denote by $\mu_B$ and $\mu_S$ the counting measure (in the finite case) or the Lebesgue measure (in the infinite case) on the sets $\mathcal{B}$ and $\mathcal{S}$.

Every order $i$ includes a value $v_i \in V = [\underline{v}, \overline{v}] \subset \mathbb{R}^+$, where $V$ is the space of possible values. We assume that the functions $v_B : \mathcal{B} \to V$ and $v_S : \mathcal{S} \to V$ that assign values to orders are Borel. A buy order's value specifies a bid, that is, the maximum willingness to pay. A sell order's value specifies an ask, that is, the minimum willingness to sell.

Given all buy and sell orders, a double auction chooses a market outcome determined by a market price $P^*$ and an allocation identifying subsets of orders,
\(B^* \subset B\) and \(S^* \subset S\), that are filled. For every filled buy order the associated trader buys one unit of the good, and for every filled sell order the associated trader sells one unit of the good. The market price is what each trader pays or receives for every filled order.

What remains to be done is to define the market price and the subsets of filled orders \(B^*\) and \(S^*\). As discussed in Section 1, the preexisting definitions relied on market finiteness or ad hoc regularity assumptions. We address this question in general, starting with the auxiliary concepts of market clearing and Walrasian equilibria.

### 3 Market clearing and Walrasian equilibria

The market price in a double auction should be market clearing, that is balance supply and demand. The terms market clearing price and Walrasian equilibrium price are often used interchangeably, albeit having different definitions. We next provide formal definitions of market clearing and Walrasian equilibrium via demand and supply, and show their equivalence.

A series of auxiliary lemmas concerning the analytical properties of demand and supply, as well as the geometrical properties of the set of market clearing prices in our framework, is relegated to Section 5. The lemmas are instructive to understand our main results but are not of particular interest in themselves, as they are known for other models (e.g., finite economies or economies with divisible goods) and therefore may not be surprising to the expert reader.\(^8\) Section 5.1 provides illustrative examples.

Consider the set of all orders with reported bid or ask (strictly) above or below a price \(P\). For a relation \(R \in \{\geq, >, =, <, \leq\}\), we introduce the shorthand notation \(B_R(P) = \{b \in B : v_b R P\}\) and \(S_R(P) = \{s \in S : v_s R P\}\).

Demand at price \(P\) is defined as the mass of all buy orders with bid above \(P\) and, similarly, supply at price \(P\) is defined as the mass of all sell orders with ask below \(P\). To account for orders with a bid or ask equal to a price \(P\), we distinguish between weak and strict inequalities by defining demand and supply

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\(^8\)For perfectly divisible goods, see, for example, Debreu (1959) and Arrow and Hahn (1971) for markets with finitely many orders and Hildenbrand (1974) for markets with a continuum of orders. For a discussion of demand and supply in the finite \(k\)-Double Auction, see, for example, Rustichini et al. (1994). For our most general case, a market with indivisible goods and a continuum of orders, see, for example, Mas-Colell (1977) and Azevedo et al. (2013) for some analytical properties of demand and supply correspondences.
functions and correspondences.

**Definition** (Demand and supply). The *demand and supply functions* at price $P$ are $D_f(P) = \mu_B(B_>(P))$ and $S_f(P) = \mu_S(S_<(P))$. The *demand and supply correspondences* are the set-valued functions $D_c(P) = [\mu_B(B_>(P)), \mu_B(B_\geq(P))]$ and $S_c(P) = [\mu_S(S_<(P)), \mu_S(S_\leq(P))]$.

Analytical properties of demand and supply functions, as well as the connection to their correspondences are relegated to Section 5.2.

Other market metrics we will use are how much trade is possible at price $P$, and how big the difference is between demand and supply at that price.

**Definition** (Trade volume and excess). The *trade volume* and *excess* at price $P$ are $Q(P) = \min(D_f(P), S_f(P))$ and $Ex(P) = |D_f(P) - S_f(P)|$. If $D_f(P) > S_f(P)$, call $Ex(P)$ *excess demand*, and if $D_f(P) < S_f(P)$, call $Ex(P)$ *excess supply*.

We now define the set of prices that balance demand and supply, that is, prices, where the demand and supply correspondences intersect.

**Definition** (Market clearing prices). A price $P$ is a *market clearing price*, if $D_c(P) \cap S_c(P) \neq \emptyset$. Denote the set of all market clearing prices by $\mathcal{P}_{MC}$.

We discuss the connection between market clearing, trade volume, and excess in Section 5.2. In Lemma 5 we observe that the set of market clearing prices is non-empty, convex, and closed. However, the excess at a market clearing price is not necessarily equal to zero.

We define Walrasian equilibrium with respect to submitted buy and sell orders. In equilibrium, the market clears, that is, the same number of buy and sell orders are filled (*trade-balance*). Every buy order with bid strictly above the market price and every sell order with ask strictly below the market price are filled (*stability*). No buy order is filled at a price above its bid price and no sell order is filled at a price below its ask price (*individual rationality*). Formally:

**Definition** (Walrasian equilibrium). A market outcome $(P^*, B^*, S^*)$ is a *Walrasian equilibrium*, if it balances trade ($\mu_B(B^*) = \mu_S(S^*)$), it is stable ($B_>(P^*) \subset \ldots$).
\( \mathcal{B}^* \) and \( \mathcal{S}_\leq(P^*) \subset \mathcal{S}^* \), and it is individually rational \( (\mathcal{B}^* \subset \mathcal{B}_\geq(P^*) \) and \( \mathcal{S}^* \subset \mathcal{S}_\leq(P^*) \).

The sets of market clearing prices and Walrasian equilibrium prices coincide.

**Proposition 1** \( \mathcal{P}_{MC} = \mathcal{P}_{EQ} \). A price is a market clearing price if and only if it is a Walrasian equilibrium price.

We provide the proof in Appendix A.

## 4 Main results

In this section we define the Double Auction and show that it results in market clearing and Walrasian equilibrium. Moreover, we show that it nests the \( k \)-Double Auction in finite markets and continuous and strictly monotone demand and supply models as special cases, and establish convergence of finite to infinite markets.

### 4.1 The Double Auction

We here present our unified mechanism of double auctions for finite and infinite markets. Recall that at the end of Section 2, we defined such auctions leaving open the question how to determine the market price and the sets of filled orders.
Definition (Double Auction: Market Price and Filled Orders).

Given all buy and sell orders, the Double Auction sets a market price in the interval of market clearing prices

\[ P^* \in [\min P_{MC}, \max P_{MC}] \]

and fills the following sets of buy orders \( B^* \) and sell orders \( S^* \):

\[ B^* = B_{\geq}(P^*) \quad \text{and} \quad S^* = S_{\leq}(P^*) \quad \text{if there is no excess at } P^* \]
\[ B^* = B_{>}(P^*) \cup \tilde{B} \quad \text{and} \quad S^* = S_{\geq}(P^*) \quad \text{if there is excess demand at } P^* \]
\[ B^* = B_{\leq}(P^*) \quad \text{and} \quad S^* = S_{<}(P^*) \cup \tilde{S} \quad \text{if there is excess supply at } P^* \]

\( \tilde{B} \subseteq B_{=}(P^*) \) and \( \tilde{S} \subseteq S_{=}((P^*) \) are sets of orders selected according to some rationing rule to ensure that the trade is balanced, that is \( \mu_B(B^*) = \mu_S(S^*) \).

The Double Auction allows for arbitrary pricing and rationing rules. Examples of pricing rules include setting the market price \( P^* \) as a convex combination of the endpoints of \( P_{MC} \) (Rustichini et al., 1994) or choosing the price uniformly at random. A standard example of a rationing rule is to select orders uniformly at random.\(^\text{10}\) We prove the existence of a uniform rationing rule in Lemma 11. Other pricing and rationing rules make the choice endogenous based on aspects of the environment that are not part of our model, e.g., based on the timing with which buy or sell orders were submitted or based on the volume of the total order.

**Theorem 1 (Properties of the mechanism).** The **Double Auction is well-defined for any market instance, it maximizes the trade volume, and implements market clearing and Walrasian equilibrium.**

**Proof.** We show in Lemma 5 in Section 5 that the set of market clearing prices \( P_{MC} \) is non-empty, compact, and convex, which implies that the interval \( [\min P_{MC}, \max P_{MC}] \) is well-defined. By Proposition 1 this set coincides with \( P_{EQ} \), proving that the Double Auction results in market clearing and Walrasian equilibrium. Lemma 9 in Section 5 proves that every market clearing price maximizes the trade volume. It is shown in the proof of Proposition 1 that an

\(^{10}\)Buy orders in \( \tilde{B} \) are selected uniformly at random from \( B_{=}(P^*) \) if the probability \( P[b \in \tilde{B}] \) is the same for all \( b \in B_{=}(P^*) \). The random selection from \( S_{=}((P^*) \) is defined analogously.
allocation exists with $S^* = S_{\leq} (P^*) \cup \tilde{S}$ and $B^* = B_{>}(P^*) \cup \tilde{B}$, where $\tilde{B} \subset B_{=}(P^*)$, $\tilde{S} \subset S_{=}(P^*)$ and at least one of the two sets is empty. \hfill \Box

\subsection*{4.2 Nesting of existing mechanisms}

For the case of finitely many orders, the standard double auction mechanism is Rustichini et al. (1994)'s formulation of the $k$-Double Auction, which extends the bilateral models of Chatterjee and Samuelson (1983) and ? and is equivalent to an earlier formulation of Wilson (1985).

\textbf{Definition (Finite $k$-Double Auction (Rustichini et al., 1994))}. Given all $m + n$ bids and asks associated to $m$ buy orders and $n$ sell orders, denoted by the set $v$, define by $v(l)$ its $l$'th smallest element. For $k \in [0, 1]$, the $k$-Double Auction sets the market price $P^*$ as

\[ P^* = kv(m) + (1 - k)v(m+1), \]

and fills all buy orders with bid strictly above and all sell orders with ask strictly below the market price to be involved in trade. For all $P^* \in (v(m), v(m+1))$, the numbers of such buy and sell orders are equal and the mechanism terminates.\textsuperscript{11} Furthermore, when $P^* = v(m)$ or $P^* = v(m+1)$, the maximum number of buy or sell orders with bid or ask equal to the market price is included in trade such that the final numbers of filled orders on both market sides are equal. In case there are more orders with bid or ask equal to the market price than can be selected, a fair lottery selects the filled orders.

In infinite markets, double auction mechanisms have been defined for textbook models with continuous and strictly monotone demand and supply functions with a unique intersection $P^*$, see, e.g., Mas-Colell et al. (1995) and Reny and Perry (2006).\textsuperscript{12}

\textbf{Definition (Continuous and strictly monotone demand and supply models (Reny and Perry, 2006; Mas-Colell et al., 1995))}. The market price $P^*$ is set as the

\textsuperscript{11}We provide a detailed discussion in Lemma 7 in Section 5.
\textsuperscript{12}In the presence of ties due to a discretization of the action space, Reny and Perry (2006, Online Supplement) provide an argument for tie-breaking in infinite markets. It can be checked straightforwardly that the tie-breaking rule is equivalent to the allocation rule of the Double Auction. However, we omit a proof, because they do not provide a full mechanism in the presence of ties.
unique intersection of the continuous and strictly monotone demand and supply functions, that is \( D_f(P^*) = S_f(P^*) \). The sets of filled buy and sell orders are \( B^* = B_\geq(P^*) \) and \( S^* = S_\leq(P^*) \) (no tie-breaking).

The Double Auction nests the \( k \)-Double Auction for finite markets and continuous and strictly monotone demand and supply models for infinite markets as special cases.

**Theorem 2** (Nesting of prior mechanisms). The Double Auction nests as special cases the \( k \)-Double Auction for finite markets (Wilson, 1985; Rustichini et al., 1994) and continuous and strictly monotone demand and supply models (Mas-Colell et al., 1995; Reny and Perry, 2006).

**Proof.** For finite markets, it is proven in Lemma 8 in Section 5.2 that the set of market clearing prices \( \mathcal{P}_{MC} \) is equal to the interval \([v^{(m)}, v^{(m+1)}]\). Therefore, the price-setting rule \( P^* = k \cdot \min \mathcal{P}_{MC} + (1 - k) \cdot \max \mathcal{P}_{MC} \) leads to the market price chosen by the \( k \)-Double Auction. Moreover, the allocation rules of the Double Auction and the \( k \)-Double Auction coincide by construction. In infinite markets with continuous and strictly monotone demand and supply functions, by assumption, there exists a market clearing price \( P^* \) with \( D_f(P^*) = S_f(P^*) \). Because demand and supply functions are continuous, Lemma 2 in Section 5.2 implies that \( D_f(P) = D_c(P) \) and \( S_f(P) = S_c(P) \) for all \( P \in V \). The strict monotonicity of demand and supply functions then implies that this market clearing price is unique, that is \( \mathcal{P}_{MC} = \{P^*\} \). This proves that the Double Auction always chooses \( P^* \) as the market price. Moreover, it holds that \( Ex(P^*) = 0 \). Therefore, by construction, the Double Auction results in the allocation \( B^* = B_\geq(P^*) \) and \( S^* = S_\leq(P^*) \), which coincides with the allocation in the textbook models of continuous and strictly monotone demand and supply.

### 4.3 Convergence

We establish convergence properties of the set of market clearing prices from finite to infinite markets. For \( k = 1, 2, 3, ... \), a sequence of finite markets \((B^k, S^k, v_B^k, v_S^k)\) is a **growing sequence of markets**, if for all \( k \geq 1 \) it holds that \( B^k \subset B^{k+1} \), \( S^k \subset S^{k+1} \), and for all \( b \in B^k \) and \( s \in S^k \) it holds that \( v_B^k(b) = v_B^{k+1}(b) \) and \( v_S^k(s) = v_S^{k+1}(s) \). We normalize measures \( \mu_B^k \) and \( \mu_S^k \), and therefore demand and supply, by multiplying with factors \( \frac{\beta}{|B^k|} \) and \( \frac{\sigma}{|S^k|} \), with \( \beta, \sigma \) strictly positive and
finite real numbers. Therefore, the measures \( \mu^k_B \) and \( \mu^k_S \) in the sequence of finite markets have fixed mass \( \beta \) and \( \sigma \), corresponding to the proportion of buy and sell orders. A growing sequence of finite markets is converge with limit \( (B, S, v_B, v_S) \) if for all \( k \geq 1 \) it holds that \( B^k \subset B, S^k \subset S, v_B^k(b) = v_B(b) \) and \( v_S^k(s) = v_S(s) \), and \( \lim_{k \to \infty} \sup_{P \in B} |D^k_f(P) - D_f(P)| = 0 \) and \( \lim_{k \to \infty} \sup_{P \in B} |S^k_f(P) - S_f(P)| = 0 \). That is, the demand and supply functions converge uniformly. Denote by \( \mathcal{P}^k_{MC} \) and \( \mathcal{P}_{MC} \) the corresponding sets of market clearing prices in the sequence of finite markets and in the limit market.

For two non-empty sets \( A, B \subset \mathbb{R} \) with representative points \( a \in A \) and \( b \in B \), we define the distance between \( a \) and \( b \) as the Euclidean distance \( d(a, b) = |a - b| \), between \( a \) and \( B \) as \( d(a, B) = \inf_{b \in B} d(a, b) \), and between \( A \) and \( B \) as \( d(A, B) = \inf_{a \in A} d(a, B) = \inf_{b \in B} d(A, b) \). Denote the diameter of a set \( A \) as \( \text{diam}(A) = |\sup A - \inf A| \).

**Theorem 3 (Convergence).** For any converge sequence of finite markets \( (B^k, S^k, v^k_B, v^k_S) \) with limit \( (B, S, v_B, v_S) \), it holds that

\[
\lim_{k \to \infty} d(\min \mathcal{P}^k_{MC}, \mathcal{P}_{MC}) = 0 \quad \text{and} \quad \lim_{k \to \infty} d(\max \mathcal{P}^k_{MC}, \mathcal{P}_{MC}) = 0.
\]

Note that by Proposition 1 the sequence of Walrasian equilibrium prices has the same convergence property. If a unique market clearing price exists in the limit, then Theorem 3 implies that \( \lim_{k \to \infty} \text{diam}(\mathcal{P}^k_{MC}) = 0 \), i.e., the set of market clearing prices becomes small in large markets.

**Proof.** To prove that \( \lim_{k \to \infty} d(\min \mathcal{P}^k_{MC}, \mathcal{P}_{MC}) = 0 \), we show that for any \( P < \min \mathcal{P}_{MC} \) there exists a market size \( k_0 \), such that for all \( k \geq k_0 \) it holds that \( \min \mathcal{P}^k_{MC} > P \). To show that \( \min \mathcal{P}^k_{MC} > P \), Lemma 4 in Section 5.2 implies that it suffices to show that there exists \( \gamma > 0 \) with \( D^k_f(P + \gamma) > S^k_f(P + \gamma) \). Because \( P < \min \mathcal{P}_{MC} \) by assumption, there exists \( \gamma > 0 \) with \( D_f(P + \gamma) > S_f(P + \gamma) \). Let \( \alpha > 0 \) be the difference of demand and supply at \( P + \gamma \), that is \( \alpha = D_f(P + \gamma) - S_f(P + \gamma) \). It follows from the uniform convergence of demand and supply, that for any \( \epsilon > 0 \) there exists \( k_0 \) such that for all \( k \geq k_0 \)

\[
|D_f(P + \gamma) - D^k_f(P + \gamma)| \leq \epsilon \quad \text{and} \quad |S_f(P + \gamma) - S^k_f(P + \gamma)| \leq \epsilon.
\]

For such \( k \), it holds that \( D^k_f(P + \gamma) - S^k_f(P + \gamma) \geq D_f(P + \gamma) - S_f(P + \gamma) - 2\epsilon = \alpha - 2\epsilon \). If
\( \epsilon > 0 \) was chosen such that \( \epsilon < \frac{\alpha}{2} \), then 
\[
D^k_f(P + \gamma) - S^k_f(P + \gamma) > 0,
\]
which shows that \( \min \mathcal{P}_{MC}^k > P \). Note that this also implies that \( \max \mathcal{P}_{MC}^k > P \). It can be proven analogously that for any \( P > \max \mathcal{P}_{MC} \) there exists a market size \( k_0 \), such that for all \( k \geq k_0 \) it holds that \( \max \mathcal{P}_{MC}^k < P \), which also implies that \( \min \mathcal{P}_{MC}^k < P \). Combining these bounds yields that \( \lim_{k \to \infty} d(\min \mathcal{P}_{MC}^k, \mathcal{P}_{MC}) = 0 \) and \( \lim_{k \to \infty} d(\max \mathcal{P}_{MC}^k, \mathcal{P}_{MC}) = 0 \).

In economic analyses of finite markets it is common to assume that agents’ preferences are drawn from continuous distributions (e.g., Myerson and Satterthwaite 1983; Chatterjee and Samuelson 1983; Rustichini et al. 1994). Are such draws likely to generate convergent sequences of economies as we grow the market size? We answer this question in the positive, thus closing the loop on the relationship between finite and infinite markets.

We study randomly generated economies and show that as the market size grows they convergent almost surely to a limit infinite market economy and we provide a bound on the speed of convergence.\(^\text{13}\) Our Theorem 4 then implies that Double Auctions in these markets also converge.

Our convergence result does not rely on any continuity assumptions on distributions from which finite economies are drawn. For simplicity and without loss of generality, we restrict our attention to equally scaled market sides, that is, we consider an infinite market \( (\mathcal{B}, \mathcal{S}, v_B, v_S) \) with \( \mu_B(\mathcal{B}) = 1 \) and \( \mu_S(\mathcal{S}) = 1 \). We construct a random growing sequence of finite markets \( (\mathcal{B}^n, \mathcal{S}^n, v^n_B, v^n_S) \) from a limit market \( (\mathcal{B}, \mathcal{S}, v_B, v_S) \) as follows: Consider a sequence of buy orders \( b_1, b_2, ..., b_n \) and a sequence of sell orders \( s_1, s_2, ..., s_n \). Note that \( P \mapsto S_f(P)/\sigma \) and \( P \mapsto 1 - D_f(P+)/\beta \) are cumulative distribution functions. The bids of buy orders and the asks of sell orders are drawn independently from distributions with these two cumulative distribution functions, which we denote by \( F_B \) for bids and \( F_S \) for asks.

**Theorem 4** (Random economies and convergence rate). A random growing sequence of finite markets \( (\mathcal{B}^n, \mathcal{S}^n, v^n_B, v^n_S) \) generated by independent sampling from a limit market \( (\mathcal{B}, \mathcal{S}, v_B, v_S) \) converges to the limit market a.s. at an exponential rate.

**Proof.** Let \( n \geq 1 \) denote the number of buy orders and the number of sell orders. The bids of buy orders are \( n \) independent random variables \( v^1_b, ..., v^n_b \)

\(^{13}\)Related insights for matching markets without transfers were established by Azevedo and Leshno (2016).
with cumulative distribution function \( F_B(P) = 1 - D_f(P^+)/\beta \). The asks of sell orders are \( n \) independent random variables \( v^1_s, ..., v^n_s \) with cumulative distribution function \( F_S(P) = s_f(P)/\sigma \). For \( A \subset V \), define \( \mu_{v,n}^B(A) = \sum_{i=1}^n \delta_{v_i}(A)/n \) and \( \mu_{v,n}^S(A) = \sum_{j=1}^n \delta_{v_j}(A)/n \). Demand and supply functions are then the random variables \( D^n_f(P) = \mu_{v,n}^B([P,v]) \) and \( S^n_f(P) = \mu_{v,n}^S([v,P]) \). The Glivenko-Cantelli Theorem implies that \( |D^n_f - D_f|_{\infty} \to 0 \) and \( |S^n_f - S_f|_{\infty} \to 0 \) almost surely. Hence, the random growing sequence of finite markets with random value measures \( \mu_{v,n}^B \) and \( \mu_{v,n}^S \) obtained by independent sampling converges almost surely to \((B, S, \mu_v^B, \mu_v^S)\).

It follows from the Vapnis-Chervonenkis Theorem,\(^{14}\) that for fixed \( \epsilon > 0 \), there exist constants \( \alpha_1, \alpha_2, \beta_1, \beta_2 > 0 \) such that \( \mathbb{P} \left[ \sup_{P \in \mathbb{R}} |D^n_f(P) - D_f(P)| > \epsilon \right] \leq \alpha_1 e^{-\beta_1 n} \), \( \mathbb{P} \left[ \sup_{P \in \mathbb{R}} |S^n_f(P) - S_f(P)| > \epsilon \right] \leq \alpha_2 e^{-\beta_2 n} \). As the bids and asks are independent, the random variables \( D^n_f \) and \( S^n_f \) are independent as well. Hence,

\[
\mathbb{P} \left[ \sup_{P \in \mathbb{R}} |S^n_f(P) - S_f(P)| > \epsilon \wedge \sup_{P \in \mathbb{R}} |S^n_f(P) - S_f(P)| > \epsilon \right] \leq \alpha e^{-\beta n}
\]

holds with \( \alpha = \alpha_1 \cdot \alpha_2 \) and \( \beta = \beta_1 + \beta_2 \). \( \square \)

## 5 Examples and auxiliary lemmas

This section contains examples that illustrate our concepts (Section 5.1) and a series of lemmas (Section 5.2). Proofs are relegated to Appendix C. The lemmas are used in the proofs of our main results.

### 5.1 Examples

In the following two finite markets we illustrate our definitions and results (including previewing some of the lemmas we introduce in the next subsection). In both markets, there are two buy orders \( B = \{b_1, b_2\} \) and one sell order \( S = \{s_1\} \) such that \( v^1_b \geq v^2_b > v^1_s \). The two markets differ in whether the buy orders include ties or not: in the market with ties \( v^1_b = v^2_b \), and in the market without ties \( v^1_b > v^2_b \).

\(^{14}\)See Vapnik and Chervonenkis (1971) and Devroye et al. (2013). The important requirement for the theorem to apply is that the Vapnik-Chervonenkis dimensions of the class of sets \( \{v \in V : v \leq P\} \) and \( \{v \in V : v \geq P\} \) are finite.
Demand and supply. Demand and supply functions are

\[ D_f(P) = \begin{cases} 
2 & \text{if } P \leq v_2^b \\
1 & \text{if } v_2^b < P \leq v_1^b \\
0 & \text{if } P > v_1^b 
\end{cases} \]

and

\[ S_f(P) = \begin{cases} 
0 & \text{if } P < v_1^s \\
1 & \text{if } P \geq v_1^s 
\end{cases} \]

For the supply correspondence, it follows from Lemma 2 that \( S_c(P) = S_f(P) \) for \( P \neq v_1^s \) and \( S_c(v_1^s) = [0, 1] \). As mentioned above, the demand correspondence depends on whether ties exist or not. Without ties, it holds that \( D_c(v_1^b) = [1, 2] \), \( D_c(v_2^b) = [0, 1] \), and \( D_c(P) = D_f(P) \) otherwise. With ties, it holds that \( D_c(v_1^b) = D_c(v_2^b) = [0, 2] \), and again \( D_c(P) = D_f(P) \) otherwise. Figure 1 illustrates both cases.

![Figure 1: Demand and supply in the examples without and with ties.](image)

(Strong) market clearing prices. In the market without ties, the set of strong market clearing prices \( P_{SMC} \) is equal to the half-open interval \( P_{MC} = (v_2^b, v_1^b] \), while the set of market clearing prices \( P_{MC} \) is equal to the closed interval \([v_1^b, v_2^b] \), see Figure 1a. Therefore, in line with Lemma 6, \( P_{MC} = P_{SMC} \) holds. In the market with ties, there exists no strong market clearing price at all, that is, the set \( P_{SMC} \) is empty. However, there exists a unique market clearing price, that is \( P_{MC} = \{v_1^b = v_2^b\} \), see Figure 1b. Therefore, again in line with Lemma 6, it holds that \( P_{MC} \) is a singleton.

Trade volume and excess. In the market without ties, any price in \((v_1^b, v_2^b]\) maximizes trade volume at 1 and minimizes excess at 0. But at \( v_1^b \), the trade excess is equal to 1 and not minimized. In the market with ties, the unique market
clearing price is \( v^1_b = v^2_b \), which maximizes trade volume at 1 and minimizes excess at 1. But every other price in \([v^1_s, v^2_b)\) has the same trade volume and excess.

**Walrasian equilibria.** In the market without ties, any price \( P \in [v^2_b, v^1_b] \) is a Walrasian equilibrium price with allocation \( B^* = \{b_1\} \) and \( S^* = \{s_1\} \). That is because this market outcome balances trade with one filled order on each market side. The allocation is individually rational, because \( v^1_b \geq P \) and \( v^1_s \leq P \), and it is stable, because \( v^2_b \leq P \). These are the only equilibrium outcomes, as any price above \( v^1_b \) or below \( v^1_s \) are not individually rational for buy order \( b_1 \) or sell order \( s_1 \), and any price in the interval \([v^1_s, v^2_b)\) leads to a market outcome that is not stable. The latter is true, because the unfilled buy order is strictly greater than \( P \). Using similar reasoning in the market with ties, we find that the price \( P = v^2_b = v^1_b \) is the unique Walrasian equilibrium price with allocation \( B^* = \{b_1\} \) or \( B^* = \{b_2\} \) and \( S^* = \{s_1\} \). In line with Proposition 1, the set of equilibrium prices coincides with the set of market clearing prices that was computed above.

**The Double Auction.** In the market without ties, the set of market clearing prices is \([v^2_b, v^1_b]\). The Double Auction selects the market price according to some pricing rule. The pricing rule \( P^* = kv^2_b + (1-k)v^1_b \) corresponds to the \( k \)-Double Auction. Note that for \( k = 0 \), the Double Auction does not minimize excess. The allocation is \( S^* = \{s_1\} \) and \( B^* = \{b_1\} \). In the market with ties, the unique market clearing price is \( P^* = v^2_b = v^1_b \). The allocation is \( S^* = \{s_1\} \) and \( B^* = \{b_1\} \) or \( B^* = \{b_2\} \) (the latter chosen by a fair coin toss). The Double Auction, therefore, coincides with the \( k \)-Double Auction.

### 5.2 Auxiliary lemmas

**Demand and supply.** Our first lemmas concern demand and supply. First, we prove that demand and supply functions are analytically well-behaved.

**Lemma 1** (Regularity of demand and supply functions). The demand function \( D_f(\cdot) \) is non-increasing, left-continuous and has right limits. The supply function \( S_f(\cdot) \) is non-decreasing, right-continuous and has left limits. The limits can be expressed as \( D_f(P+) = \mu_B(\mathcal{B}_{>}(P)) \) and \( S_f(P-) = \mu_S(\mathcal{S}_{<}(P)) \).

We then use Lemma 1 to express demand and supply correspondences.
Lemma 2 (Representation of demand and supply correspondences). It holds that $D_c(P) = [D_f(P^+), D_f(P)]$ and $S_c(P) = [S_f(P^-), S_f(P)]$. Further:

- $D_f(P) = D_c(P) \iff \mu_B(B_\pm(P)) = 0 \iff$ the demand function is continuous at $P$.
- $S_f(P) = S_c(P) \iff \mu_S(B_\pm(S)) = 0 \iff$ the supply function is continuous at $P$.

Market clearing. Next, we show that the set of market clearing prices $P_{MC}$ can be expressed in terms of demand and supply functions.\(^{15}\)

Lemma 3 ($P_{MC}$ via $D_f$ and $S_f$). $P$ is a market clearing price if and only if $D_f(P) \geq S_f(P)$ and $D_f(P'^+) \leq S_f(P')$ or $S_f(P) \geq D_f(P)$ and $S_f(P'^-) \leq D_f(P)$.

The two cases describe on which side of the market there may be excess at price $P$. For some of our results, Lemma 3 can be used to determine useful bounds on the set $P_{MC}$ in terms of demand and supply functions.

Definition (Lower and upper bounds). A price $P$ is a lower bound, if for all $P' < P$ it holds that $D_f(P') > S_f(P')$ and an upper bound, if for all $P' > P$ it holds that $S_f(P') > D_f(P')$.

Lemma 4 (Bounds on $P_{MC}$). For any price $P$:

- If $P$ is a lower bound then $P \leq \inf P_{MC}$; if additionally $P \in P_{MC}$, then $P = \min P_{MC}$.
- If $P$ is an upper bound, then $P \geq \sup P_{MC}$; if additionally $P \in P_{MC}$, then $P = \max P_{MC}$.

Next, we discuss the geometrical structure of the set of market clearing prices and, by Proposition 1, also of the set of Walrasian equilibrium prices.

Lemma 5 (Geometry of $P_{MC}$). The set of market clearing prices is non-empty, convex, and closed.

A stronger definition of market clearing that is sometimes used colloquially requires demand and supply functions to be equal at a given price.

\(^{15}\)Tatur (2005) mentions a similar construction, when demand and supply functions are given by probability distributions with jump-discontinuities.
**Definition** (Strong market clearing price). A price $P$ is a *strong market clearing price* if $D_f(P) = S_f(P)$. Denote the set of all strong market clearing prices by $\mathcal{P}_{SMC}$.

Our example in Section 5.1 shows that there might not exist a strong market clearing price. The following theorem discusses the geometric properties of the set of strong market clearing prices $\mathcal{P}_{SMC}$ and shows that the set of market clearing prices $\mathcal{P}_{MC}$ can be viewed as the minimal extension of $\mathcal{P}_{SMC}$ to guarantee existence.

**Lemma 6** (Geometry of $\mathcal{P}_{SMC}$). The set $\mathcal{P}_{SMC}$ is a convex subset of $V$. Every strong market clearing price is a market clearing price, that is $\mathcal{P}_{SMC} \subseteq \mathcal{P}_{MC}$. The set $\mathcal{P}_{MC} \setminus \mathcal{P}_{SMC}$ has Lebesgue-measure zero. Concretely, if $\mathcal{P}_{SMC} \neq \emptyset$, then $\mathcal{P}_{MC} = \mathcal{P}_{SMC}$, and if $\mathcal{P}_{SMC} = \emptyset$, then $\mathcal{P}_{MC}$ is a singleton.

In finite markets with $m$ buy orders and $n$ sell orders, Rustichini et al. (1994) show that any price in $(v^{(m)}, v^{(m+1)})$ is a strong market clearing price, hence:

**Lemma 7** (Strong market clearing). In finite markets, if $v^{(m)} \neq v^{(m+1)}$, then for $k \in (0, 1)$ the $k$-Double Auction results in a strong market clearing price. That is, $(v^{(m)}, v^{(m+1)}) \subseteq \mathcal{P}_{SMC}$.

However, if $k \in \{0, 1\}$, the $k$-Double auction might not result in a strong market clearing price.\(^{16}\) Moreover, if ties exist, the set of strong market clearing prices might be empty, see Section 5.1.\(^{17}\)

We extend Lemma 7 to solidify the folk-wisdom that the $k$-Double Auction in finite markets results in and is fully characterized by market clearing, even in the case of ties.\(^{18}\)

**Lemma 8** (Market clearing). In finite markets, for $k \in [0, 1]$, the $k$-Double Auction results in a market clearing price. Furthermore, $[v^{(m)}, v^{(m+1)}] = \mathcal{P}_{MC}$.

---

\(^{16}\)These two mechanisms, called the *seller’s* and *buyer’s Double Auction*, are often studied separately from the case $k \in (0, 1)$ (Satterthwaite and Williams, 1989; ?).

\(^{17}\)Rustichini et al. (1994); ? acknowledge that if $v^{(m)} = v^{(m+1)}$, excess might exist and the $k$-Double Auction may require rationing.

\(^{18}\)This result is already present without a formal proof in the literature. Rustichini et al. (1994) justifies it with Lemma 7, and Cripps and Swinkels (2006) states that it can be seen after “a little time with the appropriate figure” of demand and supply schedules. Both do not provide a rigorous definition of market clearing prices.
Connection of market clearing to trade volume and excess. We show that market clearing implies that trade volume is maximized, and that strong market clearing implies that excess is minimized.

Lemma 9 ($P_{MC}$ maximizes $Q(\cdot)$). A market clearing price maximizes the trade volume.

Lemma 10 ($P_{SMC}$ minimizes $Ex(\cdot)$). A strong market clearing price minimizes the excess.

Maximizing trade volume and minimizing excess are the two appealing properties from a social welfare perspective, but they are not sufficient to characterize a double auction. On the one hand, there may exist prices that maximize trade volume and minimize excess but are not market clearing. On the other hand, there also may exist two market clearing prices with different excess. Section 5.1 provides examples of such markets.

6 Conclusion

In this paper, we provide a unified definition of double auctions in finite and infinite markets. We prove that our Double Auction yields market clearing and Walrasian equilibrium, that the Double Auction nests as special cases the $k$-Double Auction for finite markets (cf. Rustichini et al. 1994) and continuous and strictly monotone demand and supply models (cf. Mas-Colell et al. 1995; Reny and Perry 2006), and we establish convergence of finite to infinite markets. Importantly, and in contrast to the previous infinite market literature, we do not impose any regularity assumptions. Such assumptions are satisfied in some but fail in other real-world applications, for example, when fees are present (Jantschgi et al., 2022).

Recently, in other settings canonical mechanisms such as deferred acceptance (Abdulkadiroğlu et al., 2015; Azevedo and Leshno, 2016), serial dictatorship (Che and Kojima, 2010), and top trading cycle (Leshno and Lo, 2020) were extended from finite to infinite markets, facilitating the analysis of these mechanisms. We hope that our unified Double Auction will have similar impact.
References


**Appendix**

Recall that $P$ is a market clearing price if and only if (i) $D_f(P) \geq S_f(P)$ and $D_f(P+) \leq S_f(P)$ or (ii) $S_f(P) \geq D_f(P)$ and $S_f(P-) \leq D_f(P)$ (Lemma 3). As a shorthand notation for our proofs, we will say that a market clearing price is of *type I*, if the first set of conditions holds, and of *type II*, if the second set of conditions hold.

**A Proof of Proposition 1**

*Proof of Proposition 1.* First, we prove that $\mathcal{P}_{MC} \subset \mathcal{P}_{EQ}$. Consider that $P^*$ is of type I, that is $D_f(P^*) \geq S_f(P^*)$ and $D_f(P^+*) \leq S_f(P^*)$. Set $S^* = S_{\leq}(P^*)$. Consider the set $\mathcal{B}_{>}(P^*)$. Lemma 1 shows that $D_f(P^+*) = \mu_B(\mathcal{B}_{>}(P^*))$. Let
\[x = S_f(P^*) - \mu_B(B_>(P^*)) \geq 0\] and let \(\tilde{B}\) be a subset of \(B_=(P^*)\) with \(\mu_B\)-measure equal to \(x\). Such a set exists because
\[
D_f(P^*) = \mu_B(B_\geq(P^*)) = \mu_B(B_>(P^*)) + \mu_B(B_=(P^*)) \geq S_f(P^*)
\]
and \(D_f(P^+*) = \mu_B(B_>(P^*)) \leq S_f(P^*)\). Set \(B^* = B_>P^* \cup \tilde{B}\). We show that \((P^*, S^*, B^*)\) is a Walrasian equilibrium. It balances trade, because \(\mu_B(B^*) = \mu_B(B_>(P^*)) + \mu_B(\tilde{B}) = \mu_B(B_>(P^*)) + S_f(P^*) - \mu_B(B_>(P^*))\). Individual rationality follows, because \(B^* \subseteq B_\geq(P^*)\) and \(S^* = S_\leq(P^*)\). It is stable, because \(B_>(P^*) \subseteq B^*\) and \(S_<(P^*) \subseteq S^*\), as \(B^* = B_>P^* \cup \tilde{B}\) and \(S^* = S_\leq(P^*)\). If there exists a market clearing price \(P^*\) of type II, that is \(S_f(P^*) \geq D_f(P^*)\) and \(D_f(P^*-) \leq D_f(P^*)\), one can construct the Walrasian equilibrium analogously.

Second, we show that \(P \notin \mathcal{P}_{MC} \Rightarrow P \notin \mathcal{P}_{EQ}\). One of two cases must hold: (i) \(D_f(P) > S_f(P)\) and \(D_f(P-) > S_f(P)\) or (ii) \(S_f(P) > D_f(P)\) and \(S_f(P-) > D_f(P)\). For (i) there exists a price \(P' > P\) with \(D_f(P') > S_f(P)\). Assume that there exist sets \(B^*\) and \(S^*\), such that \((P, B^*, S^*)\) is a Walrasian equilibrium. Individual rationality implies that \(B^* \subseteq B_\geq(P)\) and \(S^* \subseteq S_\leq(P)\) hold. Next, stability implies that \(B_>(P) \subseteq B^*\) and \(S_<(P) \subseteq S^*\) hold. Those two inclusions imply that
\[
\mu_B(B_\geq(P)) \geq \mu_B(B^*) \geq \mu_B(B_>(P)) \quad \text{and} \quad \mu_S(S_\leq(P)) \geq \mu_S(S^*) \geq \mu_S(S_<(P)).
\]

For a price \(P' > P\) it holds that \(\mu_B(B_>(P)) \geq \mu_B(B_>(P'))\). Hence,
\[
\mu_B(B^*) \geq \mu_B(B_>(P)) \geq \mu_B(B_>(P')) = D_f(P') > S_f(P) \geq \mu_S(S^*).
\]

This proves that \((P, B^*, S^*)\) is not a Walrasian equilibrium, because it does not balance trade. For (ii), the proof is analogous.

\[\square\]

**B  Uniform rationing in the Double Auction**

A standard rationing rule for Double Auctions is the uniform rationing rule. Recall that buy orders in \(\tilde{B}\) are selected uniformly at random from \(B_=(P^*)\) if the probability \(P[b \in \tilde{B}]\) is the same for all \(b \in B_=(P^*)\). The random selection from
Lemma 11 (Uniform Rationing). The uniform rationing rule is well-defined for any market instance in finite and infinite markets.

Proof. In finite markets, it suffices to show that for a discrete set $V$ with $x$ elements, it is possible to select a subset $\tilde{V}$ with $y < x$ elements, such that the probability of being selected is the same for each element. If $y = 0$, set $\tilde{V} = \emptyset$. Otherwise, order $V = \{v_1, ..., v_x\}$. Set $\tilde{V} = \{\tilde{v}_1, ..., \tilde{v}_{y-1}\}$, where $\tilde{v}_i$ is chosen uniformly at random from the set $\tilde{V}_i = V \setminus \{\tilde{v}_1, ..., \tilde{v}_{i-1}\}$ with probability $\frac{1}{x-(i-1)}$, where $\tilde{V}_1 = V$. It follows that for any $v \in V$, it holds that $P[v \in \tilde{V}] = \frac{y}{x}$.

In infinite markets, we consider a bounded Borel set $V \subset \mathbb{R}$, such that its Lebesgue measure $\lambda(V)$ is equal to $x$. We prove that for any $y < x$ it is possible to select a subset $\tilde{V}$ with Lebesgue measure $y$, such that each point of $V$ is selected with the same probability. Denote by $\overline{v}$ and $\underline{v}$ the supremum and infimum of $V$. For any Borel set, there exists a uniform probability distribution with density equal to $\frac{1}{x}$ on $V$. Using this distribution select a point $t_0$ uniformly at random. For $0 \leq \epsilon \leq \overline{v} - \underline{v}$, define the following set

$$V_{v_0}^\epsilon = \begin{cases} [v_0, v_0 + \epsilon] \cap V & \text{for } \epsilon \leq \overline{v} - v_0 \\ ([v_0, \overline{v}] \cap V) \cup ([\underline{v}, \underline{v} + \epsilon - (\overline{v} - v_0)] \cap V) & \text{for } \overline{v} - v_0 < \epsilon \leq \overline{v} - \underline{v}. \end{cases}$$

Intuitively, the set $V_{v_0}^\epsilon$ is constructed by starting at the randomly selected point $v_0$ and including all points of $V$ to its right, until the distance $\epsilon$ is reached. If the supremum of $V$ is reached, the construction proceeds at the infimum of $V$. Define the function $f_{v_0}(\epsilon) = \lambda(V_{v_0}^\epsilon)$. This function is well-defined for $\epsilon \in [0, \overline{v} - \underline{v}]$, it is nondecreasing and continuous, and it holds that $f_{v_0}(0) = 0$ and $f_{v_0}(\overline{v} - \underline{v}) = x$. It follows from the Intermediate Value Theorem, that there exists $\tilde{\epsilon}$ with $f_{v_0}(\tilde{\epsilon}) = y$. Because $f_{v_0}(\cdot)$ is non-decreasing, this set is convex. Let $\epsilon^*$ be the infimum of this set. Because $f_{v_0}(\cdot)$ is continuous, it holds that $f_{v_0}(\epsilon^*) = f_{v_0}(\lim_{\epsilon \downarrow \epsilon^*} \epsilon) = \lim_{\epsilon \downarrow \epsilon^*} f_{v_0}(\epsilon) = y$. Therefore the set $V_{v_0}^{\epsilon^*}$ is a random set with Lebesgue measure $y$. Finally, it is a straightforward computation that for all $v \in V$ it holds that $P[v \in V_{v_0}^{\epsilon^*}] = \frac{\epsilon^*}{y} \equiv \text{const.}$
C Proofs of Section 5

Proof of Lemma 1. Let $\mu^v_S$ be the pushforward measures of $\mu_S$ with respect to $v_S$, that is $\mu^v_S(\cdot) = \mu_S(v_S^{-1}(\cdot))$. $\mu^v_S$ is $\sigma$-additive and finite on $V$. Denote by $\ol{V}(P)$ the interval $[v, P]$. It holds that $S_f(P) = \mu^v_S(\ol{V}(P))$. Moreover, if $P_1 > P_2$, then $\ol{V}(P_2) \subset \ol{V}(P_1)$. The $\sigma$-additivity of $\mu^v_S$ yields

$$S_f(P_1) = \mu^v_S(\ol{V}(P_1)) \geq \mu^v_S(\ol{V}(P_2)) = S_f(P_1),$$

which proves that $S_f(\cdot)$ is non-decreasing.

Every monotonic function has limits from the right and the left for every point in its domain. Next, consider a strictly decreasing sequence of prices $P_n \downarrow P$. $\ol{V}(P_n)$ is a decreasing sequence of sets, that is $\ol{V}(P_{n+1}) \subset \ol{V}(P_n)$. It holds that $\lim_{n \to \infty} \ol{V}(P_n) = \bigcap_{n=1}^{\infty} \ol{V}(P_n) = \ol{V}(P)$. As a finite measure on $\mathbb{R}$, $\mu^v_S$ is continuous from above, see e.g., Folland (1999). That is if $\{A_i\}_i \subset V$ is a sequence of sets with $A_1 \supset A_2 \supset A_3 \supset \ldots$, then $\mu^v_S(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu^v_S(A_i)$. This yields

$$\lim_{P_n \downarrow P} S_f(P_n) = \lim_{n \to \infty} \mu^v_S(\ol{V}(P_n)) = \mu^v_S\left(\bigcap_{n=1}^{\infty} \ol{V}(P_n)\right) = \mu^v_S(\ol{V}(P)) = S_f(P),$$

which proves the right-continuity of $S_f(\cdot)$.

To show that $S_f(P-) = \mu_S(S_{<}(P)) = \mu^v_S([v, P])$, note $\mu^v_S$ is a $\sigma$-additive Borel-measure on $\mathbb{R}$ and therefore regular, see Bogachev (2007). That is, for all Borel-sets $A$ $\mu^v_S(A) = \sup \{\mu^v_S(F)|F \subset A, F \text{ compact, Borel}\}$. It is therefore sufficient to approximate the interval $[v, P]$ by compact sets $[v, P']$ with $P' < P$. It finally holds that

$$\sup \{\mu^v_S(F)|F \subset A, F \text{ compact, Borel}\} = \lim_{P' \uparrow P} \mu^v_S([v, P']) = \lim_{P' \uparrow P} S_f(P') = S_f(P-),$$

which implies that $S_f(P-) = \mu_S(S_{<}(P))$. The proof that demand is non-increasing, left-continuous, and has right limits, as well as $D_f(P+) = \mu_B(B_{>}(P))$ is analogous.

Proof of Lemma 2. Recall that $D_f(P) = \mu_B(B_{>}(P))$ and $S_f(P) = \mu_S(S_{\geq}(P))$. It follows from Lemma 1 that $D_f(P+) = \mu_B(B_{>}(P))$ and $S_f(P-) = \mu_S(S_{<}(P))$,
which implies

\[ D_c(P) = [\mu_B(\mathcal{B}_>(P)), \mu_B(\mathcal{B}_\geq(P))] = [D_f(P+), D_f(P)] \] 
\[ S_c(P) = [\mu_S(\mathcal{S}_<(P)), \mu_S(\mathcal{S}_\leq(P))] = [S_f(P-), S_f(P)]. \]

The \(\sigma\)-additivity of \(\mu_B\) implies that

\[ D_f(P) = \mu_B(\mathcal{B}_>(P)) = \mu_B(\mathcal{B}_>(P)) + \mu_B(\mathcal{B}_=(P)) = D_f(P+) + \mu_B(\mathcal{B}_=(P)). \]

Because \(D_f(P) = D_c(P)\) if and only if \(D_f(P+) = D_f(P)\), this is equivalent to \(\mu(B_\geq(P)) = 0\). Furthermore, \(D_f(P+) = D_f(P)\), that is, \(D_f\) is right-continuous at \(P\). As \(D_f(\cdot)\) is left-continuous by Lemma 1, \(D_f(P) = D_c(P)\) is equivalent to continuity at \(P\). The proof for supply is analogous. \(\square\)

**Proof of Lemma 3.** If \(D_f(P) \geq S_f(P)\), then \(D_c(P) \cap S_c(P) \neq \emptyset \iff S_f(P-) \geq D_f(P)\). If \(D_f(P) \leq S_f(P)\), then \(D_c(P) \cap S_c(P) \neq \emptyset \iff D_f(P+) \geq S_f(P)\). \(\square\)

**Proof of Lemma 4.** Consider that \(P\) is a lower bound. It suffices to prove that for a price \(P' < P\) \(P' \notin \mathcal{P}_{MC}\) holds. If \(P \in \mathcal{P}_{MC}\), it follows directly that \(P = \min \mathcal{P}_{MC}\). Because \(D_f(P') > S_f(P')\), it is sufficient to prove that \(D_f(P'') > S_f(P')\). For \(P''\) in \((P', P)\) it holds that \(D_f(P'') > S_f(P'')\). The monotonicity of \(D_f(\cdot)\) and \(S_f(\cdot)\) yields

\[ D_f(P'-) \geq D_f(P'') > S_f(P'') \geq S_f(P'). \]

If \(P\) is an upper bound, the proof is analogous. \(\square\)

**Proof of Lemma 5.** First, we show that \(\mathcal{P}_{MC}\) is non-empty. Consider the set \(\mathcal{\hat{P}} = \{P : D_f(P) \geq S_f(P)\}\). The monotonicity of \(D_f(\cdot)\) and \(S_f(\cdot)\) yields that \(\mathcal{\hat{P}}\) is convex. For \(P < \bar{v}\) it holds that \(D_f(P) = \mu_B(B) > 0 = S_f(P)\), proving that \(\mathcal{\hat{P}}\) is non-empty. For \(P > \overline{v}\) it holds that \(D_f(P) = 0 < \mu_S(S^*) = S_f(P)\), proving that \(\mathcal{\hat{P}}\) is bounded from above. For such a set the supremum \(P^* = \sup \mathcal{\hat{P}}\) exists and is unique. We show that \(P^* \in \mathcal{P}_{MC}\). Two cases need to be considered separately: (i) \(P^* \in \mathcal{\hat{P}}\) and (ii) \(P^* \notin \mathcal{\hat{P}}\).

For (i), \(D_f(P^*) \geq S_f(P^*)\) and for all \(P' > P^*\) \(D_f(P') < S_f(P')\). The right-continuity of \(S_f(\cdot)\) implies \(D_f(P^+) \leq S_f(P')\), which proves that \(P\) is a market clearing price of type I.
For (ii), \( D_f(P^*) < S_f(P^*) \) holds. The monotonicity of \( D_f(\cdot) \) and \( S_f(\cdot) \) implies that for all \( P' < P^* \) \( D_f(P') \geq S_f(P') \) holds. Left-continuity of \( D_f(\cdot) \) yields that \( D_f(P') \geq S_f(P'-) \), which proves that \( P^* \) is a market clearing price of type II. We will show below in the proof of Lemma 6 that the set of strong market clearing prices \( P_{SMC} \) is convex. The same proof proceeds to show that if \( P_{SMC} \neq \emptyset \), then \( P_{MC} = \overline{P_{SMC}} \), and if \( P_{SMC} = \emptyset \), then \( P_{MC} \) is a singleton. This proves that \( P_{MC} \) is convex and closed.

**Proof of Lemma 6.** We first show that \( P_{SMC} \) is convex. As the empty set is convex by convention, assume that \( P_{SMC} \neq \emptyset \). Consider \( P_1, P_2 \in P_{SMC} \) with \( P_1 \leq P_2 \). The monotonicity of \( D_f(\cdot) \) and \( S_f(\cdot) \) implies that \( D_f(P_1) \geq D_f(P_2) = S_f(P_2) \geq S_f(P_1) \), which proves that \( D_f(\cdot) \) and \( S_f(\cdot) \) are constant on \([P_1, P_2]\). Therefore, for any price \( P \in [P_1, P_2] \) \( P \in P_{SMC} \) holds. For \( P < \mu D_f(P) = \mu_B(B) > 0 = S_f(P) \) and for \( P > \overline{\mu D_f(P)} = 0 < \mu s(S^*) = S_f(P) \) holds, which implies that \( P_{SMC} \subset V \).

Next, we show that \( P_{SMC} \subset P_{MC} \). \( P \in P_{SMC} \iff D_f(P) = S_f(P) \). Because \( D_f(\cdot) \) is non-increasing it follows that \( D_f(P+) \geq D_f(P) = S_f(P) \), which proves that \( P \) is a market clearing price of type I. Because \( S_f(\cdot) \) is non-decreasing it follows that \( S_f(P-) \leq S_f(P) = D_f(P) \), which proves that \( P \) is a market clearing price of type II.

Next, assume that \( P_{SMC} = \emptyset \). To prove that \( P^* \) from the proof of Lemma 5 above is the unique market clearing price, it suffices to prove by Lemma 4 that \( P^* \) is both a lower and upper bound. \( P^* \) is either of type I, that is \( D_f(P^*) > S_f(P^*) \) and \( D_f(P^+ \leq S_f(P^*) \) or of type II, that is \( S_f(P^*) > D_f(P^*) \) and \( S_f(P^+ \leq D_f(P^*) \). It follows from monotonicity of \( D_f(\cdot) \) and \( S_f(\cdot) \) and the emptiness of \( P_{MC} \) that for all \( P' < P \) it holds that \( D_f(P') > S_f(P') \) and for all \( P' > P \) it holds that \( D_f(P') < S_f(P') \). Therefore \( P^* \) is indeed both a lower and upper bound.

Finally, assume that the interval \( P_{SMC} \neq \emptyset \). To show that \( P_{MC} = \overline{P_{SMC}} \), by Lemma 4 it suffices to prove that \( P = \inf P_{SMC} \) is both a market clearing price and lower bound and \( P = \sup P_{SMC} \) is both a market clearing price and upper bound. \( D_f(P) \geq S_f(P) \) by monotonicity of \( D_f(\cdot) \) and \( S_f(\cdot) \). By definition, for every \( P \) with \( P > P > P \) it holds that \( D_f(P) = S_f(P) \). It follows from the left continuity of \( S_f(\cdot) \) that \( D_f(P+) = S_f(P) \), which proves that \( P \in P_{MC} \). For every \( P' < P \) we have that \( D_f(P') > S_f(P') \). Therefore \( P \) is a lower bound. Similar arguments yield that \( P \in P_{MC} \) is an upper bound.

**Proof of Lemma 7.** We show that \( P \in (v^{(m)}, v^{(m+1)}) \Rightarrow P \in P_{SMC} \). Suppose
that \( D_f(P) = k \geq 0 \). It holds that \( D_f(P) = D_f(v^{(m+1)}) \) and \( S_f(P) = S_f(v^{(m)}) \).

The set \( \{v^{(m+1)}, \ldots, v^{(m+n)}\} \) has cardinality \( n \) and the number of bids in it is \( D_f(v^{(m+1)}) = k \). Hence, the number of asks in it is \( n - k \). As there is a total number of \( n \) asks, the number of asks in the set \( \{v^{(1)}, \ldots, v^{(m)}\} \) is \( k \). As this number is equal to \( S_f(v^{(m)}) \), it holds that \( S_f(P) = k = D_f(P) \), which shows that \( P \in \mathcal{P}_{SMC} \).

Proof of Lemma 8. There are two cases: (i) \( v^{(m)} \neq v^{(m+1)} \), and (ii) \( v^{(m)} = v^{(m+1)} \).

For (i), Lemma 7 shows that \( P \in (v^{(m)}, v^{(m+1)}) \Rightarrow P \in \mathcal{P}_{SMC} \), and hence by Lemma 6 \( P \in \mathcal{P}_{MC} \). Next, consider \( v^{(m)} \). \( S_f(v^{(m)}) = S_f(P) \) for \( P \) in \( (v^{(m)}, v^{(m+1)}) \).

If there is a bid equal to \( v^{(m)} \), \( S_f(v^{(m)}) = S_f(P) \) for \( P \) in \( (v^{(m)}, v^{(m+1)}) \). That is because there are at most \( n \) asks, the number of asks in it is \( n - k \). As this number is equal to \( S_f(v^{(m)}) \), it holds that \( S_f(P) = k = D_f(P) \). If not, then \( D_f(v^{(m)}) = D_f(P) \). This shows that \( D_f(v^{(m)}) \geq D_f(P) \). Hence,

\[
D_f(v^{(m)}) \geq D_f(P) = S_f(P) = S_f(v^{(m)}).
\]

To show that \( v^{(m)} \in \mathcal{P}_{MC} \), it is by Lemma 2 sufficient to show that \( D_f(v^{(m+1)}) \leq S_f(v^{(m)}) \). \( D_f(v^{(m+1)}) = D_f(P) \) holds, as there are no bids or asks in \( (v^{(m)}, v^{(m+1)}) \). Therefore, \( D_f(v^{(m+1)}) = D_f(P) = S_f(P) = S_f(v^{(m)}) \). A similar argument shows that \( v^{(m+1)} \in \mathcal{P}_{MC} \). Finally, we show that \( v^{(m+1)} \) is an upper bound and \( v^{(m)} \) is a lower bound, which, by Lemma 4, implies that \( v^{(m)} = \min \mathcal{P}_{MC} \) and \( v^{(m+1)} = \max \mathcal{P}_{MC} \), which finishes the proof for (i). Consider \( P > v^{(m+1)} \). \( D_f(P) < S_f(P) \) holds, as demand decreases or supply increases due to bids and asks at \( v^{(m+1)} \). Therefore \( v^{(m+1)} \) is an upper bound. Similar arguments yield that \( v^{(m)} \) is a lower bound.

For (ii), write \( v = v^{(m)} = v^{(m+1)} \) for ease of notation. We will show \( v \in \mathcal{P}_{MC} \) and that this price is both a lower and upper bound. Lemma 4 then implies that \( \mathcal{P}_{MC} = \{v\} \). For a relation \( \mathcal{R} \in \{\geq, >, =, <, \leq\} \), denote by \( v_\mathcal{R} \) the number of bids and asks (strictly) above, below or equal to \( v \). Denote by \( v_{\mathcal{R}, B} \) and \( v_{\mathcal{R}, S} \) the restriction to either bids or asks. It holds that

\[
v_\leq \leq m - 1, \ v_\geq \geq 2, \ v_\leq \leq n - 1, \ \text{and} \ v_\leq + v_\geq + v_\geq \geq m + n.
\]

Note that \( D_f(v) = v_{\geq, B} = v_{\geq, B} + v_{\geq, B} \geq 1 \). That is because there are at most \( m - 1 \) bids and asks strictly below \( v \) and there is a total of \( m \) bids, which proves that at least one bid is greater or equal to \( v \). Because the total number of asks is
\[ S_f(v) = v_{\leq S} = n - v_{> S} = n - v_+ + v_{>_B}. \]

Next, we prove that \( v \) is an upper bound. Consider \( P > v \). \( D_f(P) \leq v_{>_B} \) holds, because \( v_{=B} \) bids at \( v \) are lost and \( S_f(P) \geq n - v_+ + v_{>_B} \), because supply is non-decreasing. Hence, \( S_f(P) - D_f(P) \geq n - v_+ \). \( v_+ \leq n - 1 \) implies that \( S_f(P) - D_f(P) \geq 1 \), which yields \( S_f(P) > D_f(P) \).

Finally, we prove that \( v \) is a lower bound. Consider \( P < v \). It holds that \( S_f(P) \leq n - v_+ + v_{>_B} - v_+ + v_{=B} \), because \( v_+ - v_{=B} \) asks at \( v \) are lost and \( D_f(P) \geq v_{=B} + v_{>_B} \), because demand is non-increasing. This implies that \( D_f(P) - S_f(P) \geq v_+ + v_{>_B} - n \). But it follows from \( v_+ + v_{=B} + v_{>_B} = m + n \) and \( v_+ \leq m - 1 \) that \( v_+ + v_{>_B} = m + n - v_+ \geq n + 1 \), which implies that \( D_f(P) - S_f(P) \geq 1 > 0 \). Therefore, \( D_f(P) > S_f(P) \).

**Proof of Lemma 9.** Consider \( P \in \mathcal{P}_{MC} \). Assume that \( P \) is of type I, that is, \( D_f(P) \geq S_f(P) \) and \( D_f(P^+) \leq S_f(P) \). Then \( Q(P) = \min(D_f(P), S_f(P)) = S_f(P) \) holds. For any price \( P' < P \), \( S_f(P') \leq S_f(P) \) holds, because \( S_f(\cdot) \) is non-decreasing. Therefore \( Q(P') \leq Q(P) \). For any price \( P'' > P \), it holds that \( D(P'') \leq D(P^+) \), because \( D_f(\cdot) \) is non-increasing. \( S_f(P) \geq D_f(P^+) \) implies that \( S_f(P) \geq D_f(P'') \). Therefore \( Q(P'') \leq Q(P) \), which proves that \( P \) maximizes the trade volume. The proof for a market clearing price of type II proceeds in analogy.

**Proof of Lemma 10.** \( P \in \mathcal{P}_{SMC} \iff D_f(P) = S_f(P) \) and hence \( Ex(P) = 0 \).